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TITLE: **HOMOTOPY QUANTUM FIELD THEORY AND QUANTUM GROUPS**

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# HOMOTOPY QUANTUM FIELD THEORY AND QUANTUM GROUPS

by

**Neha Gupta**

**Thesis**

Submitted to the University of Warwick

for the degree of

**Doctor of Philosophy**

*Supervisors:* Dr Dmitriy Rumynin

Department of Mathematics

February 2011

THE UNIVERSITY OF

WARWICK

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# Contents

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# Declaration

I declare that, to the best of my knowledge, the material contained in this thesis is original and my own work, except where otherwise indicated, cited, or commonly known. The material in this thesis is submitted for the degree of Ph.D. to the University of Warwick only, and has not been submitted to any other university.

# Abstract


The thesis is divided into two parts one for dimension 2 and the other for dimension 3.

Part one (Chapter 3) of the thesis generalises the definition of an $n$-dimensional HQFT in terms of a monoidal functor from a rigid symmetric monoidal category $\mathcal{X} - \mathrm{Cob}_n$ to any monoidal category $\mathcal{A}$. In particular, 2-dimensional HQFTs with target $K(G, 1)$ taking values in $\mathcal{A}$ are generated from any Turaev $G$-crossed system in $\mathcal{A}$ and vice versa. This is the generalisation of the theory given by Turaev into a purely categorical set-up.

Part two (Chapter 4) of the thesis generalises the concept of a group-coalgebra, Hopf group-coalgebra, crossed Hopf group-coalgebra and quasitriangular Hopf group-coalgebra in the case of a group scheme. Quantum double of a crossed Hopf group-scheme coalgebra is constructed in the affine case and conjectured for the more general non-affine case. We can construct 3-dimensional HQFTs from modular crossed $G$-categories. The category of representations of a quantum double of a crossed Hopf group-coalgebra is a ribbon (quasitriangular) crossed group-category, and hence can generate 3-dimensional HQFTs under certain conditions if the category becomes modular. However, the problem of systematic finding of modular crossed $G$-categories is largely open.

# Chapter 1

# Introduction

Birth and development of a new fascinating mathematical theory has taken place in the past couple of decades. Algebraists often call it the theory of quantum groups whereas topologists prefer calling it quantum topology. This new field of study is mostly associated to the already known theory of Hopf algebras, the theory of representations of semisimple Lie algebras, the topology of knots etc. The most phenomenal achievements in this theory are centred around quantum groups and invariants of knots and 3-dimensional manifolds. perhaps, the whole theory has been motivated by the intellection that arose in theoretical physics. The evolution, growth and development of this new subject has once again proved that physics and mathematics are so well interconnected and interrelated. They often influence each other to the betterment of both disciplines.

A brief historical background is important for a better understanding of this new subject. The introduction of a new polynomial invariant of classical knots and links by Jones,V. (1984) blazes the history of this theory. Quantum groups were introduced in 1985 by V.Drinfeld and M.Jimbo, which may be broadly described as 1-parameter deformations of semisimple complex Lie algebras. Note that quantum groups transpired as a as an algebraic formalism for the philosophies given out by physicists, specifically, from the work of the Leningrad school of mathematical physics directed by L.Faddeev.

In 1988, E.Witten invented the notion of a Topological Quantum Field Theory (TQFT) and characterised an intriguing picture of such a theory in three dimensions. The most important contributions towards the development of the subject (in its topological part) has been mainly influenced by the works of people like M.Atiyah, A.Joyal, R.Street, L.Kauffman, A. Kirillov, N.Reshetikhin, G.Moore, N.Seiberg, N.Reshetikhin, V.Turaev, G.Segal and O.Viro.

We start our discussion with a quantum field theory in general. Roughly, a quantum field theory takes as input *spaces* and *space-times* and associates to them *state spaces* and *time* evolution *operators*. The space is modelled as a closed oriented $(n-1)$-manifold, while space-time is an oriented $n$-manifold whose boundary represents time 0 and time 1. The state space is a vector space (over some ground field $\mathbb{K}$), and the time evolution operator is simply a linear map from the state space of time 0 to the state space of time 1. The theory is called *topological* if it only depends on the topology of the space-time and independent of energy. This means that 'nothing happens' as long as time evolves cylindrically.

Though the main abstraction of a topological quantum field theory (TQFT) is influenced by the work of E.Witten, [Wit88], [Wit89]; its axiomatic analogue was first formulated by Atiyah, extending G.Segal's axioms for the modular functor. Roughly speaking a topological quantum field theory (TQFT) in the axiomatic setting, in dimension $n$ defined over a ground ring $K$, consists of the following data: (i) A finitely generated $K$-module $Z(S)$ associated to each oriented closed smooth $n$-dimensional manifold $S$, (ii) An element $Z(M) \in Z(\partial M)$ associated to each oriented smooth $(n+1)$-dimensional manifold (with boundary) $M$. These data are subject to the axioms requiring $Z$ to be functorial, involutory and multiplicative ([Ati88]).

In 1991, Quinn carried out a systematic study of axiomatic foundations of TQFTs in an abstract set up. In his lecture notes [Qui95] Quinn has further made annotations for the definition of TQFTs in the the categorical setting. Quinn also takes the opportunity to generalise the whole setting: in his definition a TQFT does not only talk about cobordisms, but more generally about a domain category for TQFT which is a pair of categories related by certain functors and operations which play the role of space and space-time categories in the usual cobordism settings. A functorial definition of a TQFT simply says that a TQFT is a symmetric monoidal functor from the domain category (category of cobordisms) to the category of vector spaces over a ground field $\mathbb{K}$.

It may be noted that the notion of a monoidal (tensor) category was introduced by Saunders Mac Lane [Mac63]. Then duality in monoidal categories has been discussed by several authors like [KL80], [FY89], [JS91]-[JS93]. A braiding in a monoidal category was formally defined first by Joyal and Street, [JS93].

In the year 1999, Turaev generalised the idea of a topological quantum field theory to maps from manifolds into topological spaces. This leads to a notion of a $(n+1)$-dimensional Homotopy Quantum Field Theory(HQFT) which may be described as a version of a TQFT for closed oriented $n$-dimensional manifolds and compact oriented $(n+1)$-dimensional cobordisms endowed with maps into a fixed topological space $X$. Such an HQFT yields numerical homotopy invariants of maps from closed oriented $(n+1)$-dimensional manifolds to $X$. Hence a TQFT may be interpreted in this language as an HQFT with target space consisting of one point. The general notion of a $(n+1)$-dimensional HQFT was introduced by Turaev in [Tur99]. From a wider prospective, Homotopy Quantum Field Theory (HQFT) is a branch of Topological Quantum Field Theory founded by E. Witten and M. Atiyah. It applies ideas from theoretical physics to study principal bundles over manifolds and, more generally, homotopy classes of maps from manifolds to a fixed target space. The first systematic account of an HQFT has been analysed in the book "Homotopy Quantum Field Theory" with appendices by Michael Muger and Alexis Virelizier, [Tur10a]. The book starts with a formal definition of an HQFT and provides examples of HQFTs in all dimensions. The main body of the text in the book is focused on 2-dimensional and 3-dimensional HQFTs. The study of these physics-oriented and topologically interpreted inventions (like TQFTs, HQFTs, etc.) lead to new algebraic objects: crossed Frobenius group-algebras, crossed ribbon group-categories, and Hopf group-coalgebras. These notions and their connections with HQFTs are discussed in detail in the book.

Given that the ground field is $\mathbb{K}$, the $(0+1)$-dimensional HQFTs with target $X$ correspond bijectively to finite-dimensional representations over $\mathbb{K}$ of the fundamental group of $X$ or equivalently, to finite-dimensional flat $\mathbb{K}$-vector bundles over $X$. This allows one to view HQFTs as high-dimensional generalisations of flat vector bundles. Turaev, [Tur99], has studied algebraic structures underlying such HQFTs when the target space is the

Eilenberg-MacLane space $K(\pi, 1)$ for a (discrete) group $\pi$. For $n = 1$, these structures are formulated in terms of $\pi$-graded algebras. A $\pi$-graded algebra is an associative unital algebra $L$ endowed with a decomposition $L = \bigoplus_{\alpha\in\pi} L_\alpha$ such that $L_\alpha L_\beta \subseteq L_{\alpha\beta}$ for any $\alpha, \beta \in \pi$. The $\pi$-graded algebra (or simply, $\pi$-algebra) arising from (1+1)-dimensional HQFTs have additional features including a natural inner product and an action of $\pi$. This led to a notion of a crossed Frobenius $\pi$-algebra. Turaev's main result concerning (1+1)-dimensional HQFTs with target $K(\pi, 1)$ is the establishment of a bijective correspondence between the isomorphism classes of such HQFTs and the isomorphism classes of crossed Frobenius $\pi$-algebras. This generalises the standard equivalence between (1+1)-dimensional TQFTs and commutative Frobenius algebras (the case $\pi = 1$). Thus he has characterised (1+1)-dimensional HQFTs whose target space is the space $K(\pi, 1)$. He has classified the (1+1)-dimensional HQFTs in terms of crossed group-algebras. His second main result is the classification of semisimple crossed Frobenius $\pi$-algebra in terms of (1+1)-dimensional cohomology classes of the subgroups of $\pi$ of finite index.

At about the same time, Brightwell and Turner (1999) looked at what they called the homotopy surface category and its representations. In the 2-dimensional case, the notion of an HQFT was introduced independently of Turaev [Tur99], by M.Brightwell and P.Turner [BTW03]. These authors classified 2-dimensional HQFTs with simply connected targets in terms of Robenia's algebras. The role of 2-categories in this setting was discussed in their subsequent paper [BT03]. Relative 2-dimensional HQFTs with target $X = \big(K(G, 1), x\big)$ were introduced and studied by G.Segal and G.Moore. Some new geometric proofs of a few theorems first established by [Tur99] has also been discussed by the authors.

There are two different viewpoints which interact and complement each other. The point of view constituted by Turaev seems to be to look at HQFTs as an extension of the toolkit for studying manifolds given by TQFTs. On the other hand, in the viewpoint of Brightwell and Turner, it is the background space, which is interrogated by the surfaces in the sense of sigma-models.

The axiomatic definition of HQFTs introduced in [Tur99] was analysed and improved by G.Rodrigues [Rod03]. A related notion of a homological quantum field theory was introduced by E.Castillo and R.Diaz [CD05].

A fundamental connection between 1-dimensional quantum field theories and braided crossed $G$-categories has been established by Muger, [Müg05]. He has shown that a quantum field theory on the real line having a group $G$ of inner symmetries brings out a braided crossed $G$-category (the category of twisted representations). Its neutral subcategory is equivalent to the usual representation category of the theory.

In 2000, Turaev came up with his new work on (2+1)-dimensional HQFTs. He has discussed in detail the 3-dimensional HQFTs with target space $K(\pi, 1)$, [Tur00]. A manifold $M$ endowed with a homotopy class of maps $M \to K(\pi, 1)$ is called a $\pi$-manifold. The homotopy classes of maps $M \to K(\pi, 1)$ classify principal $\pi$-bundles over $M$ and (for connected $M$) bijectively correspond to the homomorphisms $\pi_1(M) \to \pi$. His approach to 3-dimensional HQFTs is based on a connection between braided categories and knots. This connection plays a key role in the construction of topological invariants of knots and 3-manifolds from quantum groups. In this paper he has instituted an algebraic technique allowing to construct 3-dimensional HQFTs.

Starting from a $\pi$-category, he introduced, for a group $\pi$, the notion of a crossed $\pi$-category. Examples of $\pi$-categories can be set up from the so-called Hopf $\pi$-coalgebras. The notion of a Hopf $\pi$-coalgebra generalises that of a Hopf algebra. Similarly, the notion of a crossed Hopf $\pi$-coalgebra generalises that of a crossed Hopf algebra, which is a Hopf algebra equipped with an action of the group $\pi$ by Hopf algebra automorphisms. He studied braidings and twists in such categories which led him to lay the notion of modular crossed $\pi$-categories. He showed that each modular crossed $\pi$-category gives rise to a three-dimensional HQFT with target $K(\pi, 1)$. This HQFT has two ingredients: a "homotopy modular functor" $A$ assigning projective $\mathbb{K}$-modules to $\pi$-surfaces and a functor $\tau$ assigning $\mathbb{K}$-homomorphisms to 3-dimensional $\pi$-cobordisms. In particular, the HQFT provides numerical invariants of closed oriented 3-dimensional $\pi$-manifolds. For $\pi = 1$, one recovers the familiar construction of 3-dimensional TQFTs from modular categories. Turaev has discussed various algebraic methods of producing crossed $\pi$-categories. He has also shown how crossed $\pi$-categories arises from quasitriangular Hopf $\pi$-coalgebras. However, the problem of systematic finding of modular crossed $\pi$-categories is mostly unexplored.

A braided $\pi$-categories, also called $\pi$-equivariant categories, determines a algebraic analogue for orbifold models that arise in the study of conformal field theories where $\pi$ is the group of automorphisms of the vertex operator algebra, see Kirillov (2004). The category of representations of a quasitriangular Hopf $\pi$-coalgebra provides an example of a braided $\pi$-category, see Turaev [Tur00], A.Virelizier [Vir02].

Hopf group ($\pi$)-coalgebras were studied by Turaev and further investigated by A.Hegazi and co-authors [AM02], [HIE08] and by S.H.Wang: [Wan04a], [Wan04b], [Wan04c], [Wan07], [Wan09]. Note that these Hopf group coalgebra structures are a generalisation of coloured Hopf algebras which were introduced by Ohtsuki, [Oht93]. In particular, when $\pi$ is abelian, one recovers a coloured Hopf algebra from a Hopf $\pi$-coalgebra.

Ohtsuki introduced Hopf algebra, quasi Hopf algebra, ribbon Hopf algebra and universal $R$-matrices in coloured version. He laid the foundation of these algebra structures to retrieve the invariants of knots and links. Many people defined various invariants of links, and it appeared that most of these invariants can be obtained via representations of quantum groups $U_q(\mathfrak{g})$. There are two procedures to get polynomial invariants of links extracted from $U_q(\mathfrak{g})$. The first procedure is to use the parameter $q$ of $U_q(\mathfrak{g})$; for example, one gets Jones polynomial. The other is to deform a representation of $U_q(\mathfrak{g})$, for example polynomial invariants which are essentially the deformation parameters of representations. Ohtsuki [Oht93] defines universal invariants of links which proved to be quite helpful to put together the invariants formulated from various representations of $U_q(\mathfrak{g})$. He gives explicit formulation of Universal $R$-matrices for coloured representations of $U_q(sl_2)$ by deforming quotients of $U_q(sl_2)$.

A categorical approach to Hopf $\pi$-coalgebras was introduced by S.Caenepeel and M.DeLombaerde [S.C04]. Later on these algebraic gadgets have been used by Virelizier to construct Hennings-like and Kuperberg-like invariants of principal $\pi$-bundles over link complements and over 3-manifolds. A generalisation of Hopf $\pi$-coalgebras to so-called $\pi$-cograded multiplier Hopf algebras was established by A.Van Daele, L.Delvaux and their co-authors [AEHDVD07], [Del08], [DVD07], [DvDW05].

A. Virelizier [Vir05] has worked out some non-trivial examples of quasitriangular Hopf

$\pi$-coalgebras with finite dimensional components. He restricted to a less general situation: the initial datum is not any crossed Hopf $\pi$-coalgebra, but a Hopf algebra endowed with an action by Hopf algebra automorphisms. It is worth to point out that the component $H_1$ of a Hopf $\pi$-coalgebra $H = \{H_\alpha\}_{\alpha\in\pi}$ is a Hopf algebra and that a crossing for $H$ induces an action of $H_1$ on $H_1$ by Hopf automorphisms making $H_1$ a crossed Hopf algebra in the usual sense.

The notion of a braiding in a monoidal category was introduced by Joyal and Street [JS91]-[JS93]. The definition of a twist in a braided category was given by Shum [Shu94]. These authors use the term balanced tensor category for a monoidal category with braiding and twist, and the term tortile tensor category for a monoidal category with braiding, twist, and compatibility duality. It is Turaev, [Tur08] who leads into Braided crossed $G$-categories. He came up with the notion of these categories on the basis of a study of representations of the quantum group $U_q(\mathfrak{g})$ at roots of unity. Analogues of the Yetter-Drinfeld modules in the context of braided crossed $\pi$-categories were studied by F.Panaite and M.Staic [PS07].

In the theory of Hopf algebras the structures parallel to a braiding and a twist were introduced by Drinfel'd [Dri85], [Dri87], Jimbo [Jim85], and Reshetikhin and Turaev [RT90]. Hopf algebras with these structures are called quasitriangular and ribbon Hopf algebras.

Algebraic properties and topological applications of crossed Hopf $\pi$-coalgebras has been systematically studied by A. Virelizier, [Vir02]. In this papaer, he has shown the existence of integrals and traces for such coalgebras and generalised to them the crucial properties of a usual quasitriangular and ribbon Hopf algebra.

M. Zunino [Zun04a] generalised the Drinfeld quantum double of a Hopf algebra to a crossed Hopf $\pi$-coalgebra. He constructed, for a crossed Hopf $\pi$-coalgebras $H = \{H_\alpha\}_{\alpha\in\pi}$, a double $Z(H) = \{Z(H)_\alpha\}_{\alpha\in\pi}$ of $H$, which is a quasitriangular crossed Hopf $\pi$-coalgebra in which $H$ is embedded. One has that $Z(H)_\alpha = H_\alpha \otimes (\bigoplus_{\beta\in\pi} H^*_\beta)$ as a vector space. Unfortunately, each component $Z(H)_\alpha$ is infinite-dimensional (unless $H_\beta = 0$ for all but a finite number of $\beta \in \pi$). He showed that if $\pi$ is finite and $H$ is semisimple, then $Z(H)$ is modular. In his paper, Zunino also defined a double for crossed $\pi$-categories and established its compatibility with representation theory: for a crossed Hopf $G$-coalgebra $H$, the representation category of $D(H)$ is equivalent to the double of the representation category of $H$. Symbolically, $\operatorname{Rep} D(H) \approx D(\operatorname{Rep} H)$. This shows that Zuninos double keeps the main features of the Drinfeld double.

The thesis is divided mainly into two parts- part one is for dimension 2 and part two is for dimension 3.

Part one of the thesis generalises the definition of an $n$-dimensional HQFT in terms of a monoidal functor from $\mathcal{X}-\mathrm{Cob}_n$ to any monoidal category $\mathcal{A}$. In particular, 2-dimensional HQFTs with target $K(G,1)$ taking values in $\mathcal{A}$ are generated from any Turaev $G$-crossed system in $\mathcal{A}$ and vice versa. This is the generalisation of [Tur99] into a purely categorical set-up.

Part two of the thesis generalises the concept of a group-coalgebra, Hopf group-coalgebra, crossed Hopf group-coalgebra and quasitriangular Hopf group-coalgebra in the case of a group scheme. Quantum double of a crossed Hopf group-scheme coalgebra is constructed in the affine case and conjectured for the more general non-affine case. We can construct 3-dimensional HQFTs from modular crossed $G$-categories, [Tur00]. The category of representations of a quantum double of a crossed Hopf group-coalgebra is a ribbon (quasitriangular) crossed group-category, and hence can generate 3-dimensional HQFTs under certain conditions if the category becomes modular. However, the problem of systematic finding of modular crossed $G$-categories is largely open.

More elaborately, there are three important developments integrated in this thesis, and all the original results are encapsulated in chapters 3 and 4. In the first development, we try to generalise the concepts of a group algebra/coalgebra, Frobenius group-coalgebra and finally a crossed group-coalgebra (given by Turaev) into a categorical set-up. Throughout this part of the thesis, we fix a discrete multiplicative group $G$ with identity element as $e$ and $\mathcal{C}$ a monoidal category. We start by discussing definitions of $G$-coalgebra and a $G$-algebra structures in any such monoidal category $\mathcal{C}$. Then, we develop the theory of Frobenius extensions in monoidal category $\mathcal{C}$. The three equivalent characterisations of a Frobenius extension in such a category is discussed in the form of a small result. We then go on further to define a Frobenius graded system. A similar characterisation in the graded case is also analysed.

Inspired by the work done by Turaev, [Tur10a] on Homotopy Quantum Field Theories, we define a Turaev crossed $G$-system which is a generalisation of a crossed group coalgebra defined in the category of $\mathbb{K}$-vector spaces where $\mathbb{K}$ is the ground field. In the last of the first part we construct a few examples of a Turaev crossed system. Given a crossed module $(H, \pi, t, u)$, we formulate a Turaev crossed $\pi$-system in the category of $\mathbb{K}$-modules where $\mathbb{K}$ is a commutative ring with unit. For another set of example, we first construct a twisted category $\mathcal{A}_\pi^{\sigma,\tau}$ where $(\sigma, \tau)$ is an abelian 3-cocycle of a group $\pi$ with values in $\mathbb{K}^\times$. Then using the abelian 3-cocycle, we produce another Turaev crossed $\pi$-system but now in $\mathcal{A}_\pi^{\sigma,\tau}$.

In the past, an interesting connection between the notion of Frobenius algebra or the more general Frobenius extension on the one hand and 2-dimensional topological quantum field theories on the other hand has been established. Recently, Turaev has defined so called HQFTs and laid the connection between 2-dimensional HQFTs and crossed group coalgebras. We try to generalise this concept which marks the second development of the thesis. For doing this, we first construct a symmetrical monoidal category $\mathcal{X} - \mathrm{Cob}_n$ in degree $n$, where $\mathcal{X} = (X, x)$ is a pointed path-connected topological space. This is done in three steps. First we frame a weak 2-category $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$. It is weak in two senses. First, the associativity and the identity properties of compositions of 1-morphisms holds only up to a 2-isomorphism. Second, the composition of 2-morphisms is not associative either, although one could make it associative up to a 3-morphism by turning $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$ into a 3-category. The weak 2-category $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$ plays an auxiliary role and its exact axioms are of no significance for the further discussion.

By a manifold we understand a compact oriented topological manifold with boundary. A closed manifold would mean a manifold in the above sense but now without boundary. The dimension of a manifold is the dimension of any of its components that must be equal for the dimension to exist.

An object is a triple $\mathcal{M} = (M, f, p)$ where $M$, called the *base space* of $\mathcal{M}$, is a closed manifold of dimension $n$, such that every component of $M$ is a pointed closed oriented manifold, $f : M \to X$ is a continuous function and $p$ is a point on each component of

$M$. The continuous function $f$, called as the *characteristic map* of $M$, is required to be a morphism of pointed manifolds, that is, $f(p(X)) = x$ for any $X \in \pi_0(M)$. That is to say it sends the base points of all the components of $M$ into $x$. A morphism from $\mathcal{M} = (M, f_M, p_M)$ to $\mathcal{K} = (K, f_K, p_K)$ is a triple $\mathcal{A} = (A, f_A, \alpha_A)$ where $A$, called the *base space* of $\mathcal{A}$, is a manifold of dimension $n+1$, $f_A : A \to X$ is a continuous map, called characteristic map of $A$, and $\alpha_A : \partial A \to (-M) \sqcup K$, called the *boundary map* of $A$, is an $\mathcal{X}$-homeomorphism. $A$ also has a canonical map $p_A : \pi_0 : (\delta A) \to A$ referred to as pointed structure on the boundary of $A$. Finally, we define 2-morphisms in $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$ as homotopies up to an isotopy on the boundary. Let us spell it out. Consider two 1-morphisms $\mathcal{A}, \mathcal{B} : \mathcal{K} \to \mathcal{M}$. A 2-morphism $\Phi : \mathcal{A} \to \mathcal{B}$ is a triple $(\phi, \alpha, \gamma)$ where $\phi : A \times [0,1] \to X$ and $\alpha : \partial A \times [0,1] \to (-K) \sqcup M$ are continuous maps such that :

(i) $(A, \phi_0, \alpha_0) = \mathcal{A}$,

(ii) $(A, \phi_t, \alpha_t)$ is a 1-morphism from $\mathcal{K}$ to $\mathcal{M}$ for each $t \in [0,1]$.

(iii) $\gamma : (A, \phi_1, \alpha_1) \to \mathcal{B}$ is an $\mathcal{X}$-homeomorphism of 1-morphisms.

The composition of 2-morphisms is defined by cutting the interval $[0,1]$ in half. This composition is associative only up to homotopy (on $[0,1]$). This is where 3-morphisms appear! Similarly the trivial homotopy is identity 2-morphism only up to homotopy and each 2-morphism admits an inverse up to homotopy. One can fix this by choosing homotopy classes of 2-morphisms but we do not do it because our interest in 2-morphisms is temporary. We say that two 1-morphisms $\mathcal{A}, \mathcal{B} : \mathcal{K} \to \mathcal{M}$ are *equivalent* if there exists a 2-morphism from $\mathcal{A}$ to $\mathcal{B}$ and in that case we write $\mathcal{A} \sim \mathcal{B}$. We say 0-morphisms $\mathcal{M} = (M, f_M, p_M)$ and $\mathcal{K} = (K, f_K, p_K)$ are *isomorphic* if there are 1-morphisms $\mathcal{A} = (A, f_A, \alpha_A)$ and $\mathcal{B} = (B, f_B, \alpha_B)$ from $\mathcal{M}$ to $\mathcal{K}$ and $\mathcal{K}$ to $\mathcal{M}$ respectively, such that $I_{\mathcal{K}} \sim \mathcal{A} \circ \mathcal{B}$ and $I_{\mathcal{M}} \sim \mathcal{B} \circ \mathcal{A}$. In this case we say $\mathcal{B}$ is an inverse of $\mathcal{A}$. This formulates a weak 2-category $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$. Second, we define an intermediate category $\mathcal{X} - \widehat{\mathrm{Cob}}_n$. Its objects are 0-morphisms of $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$. Its morphisms are equivalence classes of 1-morphisms in $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$. Refer section (3.4.4) for definition of a cylinder that we will be using in the thesis. Finally, for each connected isomorphism class in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ we choose a representative. Let $\mathcal{X} - \mathrm{Cob}_n$ be a full subcategory of $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ whose objects are closed under taking disjoint unions of these chosen representatives. We call this category a *cobordism category* of dimension $n$. Indeed, $\mathcal{X}-\mathrm{Cob}_n$ is a symmetric monoidal category $\mathcal{C}$. Now, being equipped with such a category $\mathcal{X}-\mathrm{Cob}_n$, we relate an $\mathcal{X}$-HQFT, $(\mathcal{Z},\tau)$ as a monoidal functor from $\mathcal{X}-\mathrm{Cob}_n$ to any symmetric monoidal category $\mathcal{C}$. And in this case, we say that the $\mathcal{X}$-HQFT $(\mathcal{Z},\tau)$ takes value in $\mathcal{C}$ (for details see chapter-2).

We further show that 0-morphisms in $\mathcal{X}-\mathrm{Cob}_1$ (we call them circles) form a Frobenius system for any $X = K(G,1)$ space. Moreover, cylinders form a Turaev $G$-crossed system in $\mathcal{X}-\mathrm{Cob}_1$.

Note that in [GTMW09], the authors have constructed an embedded cobordism category which generalises the category of conformal surfaces, introduced by G. Segal. They have identified the homotopy type of the classifying space of the embedded $d$-dimensional cobordism category for all $d$. The spirit in introducing the cobordism category is the same, but they do it quite differently. The objects in their $d$-dimensional cobordism category are closed $(d-1)$-dimensional smooth submanifolds of high-dimensional euclidean space and the morphisms are $d$-dimensional embedded cobordisms with a collared boundary.

The last part(development) of the thesis is mainly inspired by the work of Zunino [Zun04a] and the work of Virelizier [Vir02] and [Vir05]. In this part we mainly work with algebraic groups ($\mathbb{G}$) and group schemes ($\mathcal{G}$). Recall that in the first part of the thesis, we try to generalise group coalgebra and group algebra in a categorical set-up. As in this part we are mainly working with group schemes we generalise these concepts into a group scheme coalgebra and a group scheme algebra. We then define a Hopf $\mathcal{G}$-coalgebra and discuss a crossed structure on it. This work is inspired by Turaev, [Tur99]. Further we shall discuss quasitriangular structures on a Hopf $\mathcal{G}$-coalgebra. Finally we construct quantum double of a Hopf $\mathcal{G}$-coalgebra. This part of the thesis is inspired by the work of Zunino, [Zun04a].

We shall work with a group scheme $\mathcal{G}$ over the ground field $\mathbb{K}$. We think of it as a Zariski topological space $\mathcal{G}$ together with a structure sheaf of algebras $\mathcal{O}_{\mathcal{G}}$ on $\mathcal{G}$.

The multiplication is $\mu : \mathcal{G}\times\mathcal{G}\to\mathcal{G}$, the inverse is $\iota : \mathcal{G}\to\mathcal{G}$, the identity is $e : p\to\mathcal{G}$, where $p$ is the spectrum of $\mathbb{K}$ (the point) and the conjugation is $c : \mathcal{G}\times\mathcal{G}\to\mathcal{G}$ such that $(g,h)\mapsto hgh^{-1}$ for $g,h\in\mathcal{G}$. We will denote this action by ${}^{h}g\mapsto hgh^{-1}$. Note that

$e \in \mathcal{G}(\mathbb{K})$. If we denote $1_R \in \mathcal{G}(R)$ as the unit element of each group $\mathcal{G}(R)$ for a $\mathbb{K}$-algebra $R$, then $e = 1_{\mathbb{K}}$ is the unit of $\mathcal{G}(\mathbb{K})$.

The quasitriangular structures on a Hopf $\mathcal{G}$-coalgebra is discussed at the global level but the axioms to be satisfied by the universal $R$-matrix is better understood at a specialisation of $\mathcal{G}$ at some commutative ring $\mathbb{S}$. Then working at level of fibres one recovers the corresponding definitions as in the case of a discrete group, [Vir02].

Quantum double is discussed and defined in the affine case and conjectured in the case of a general group scheme.Though this part is entirely inspired by the work done by Zunino, [Zun04a], but there is a subtle difference. He requires his Hopf group coalgebra, $H = \{H_\alpha\}_{\alpha\in\pi}$ (what he calls as a Turaev-coalgebra) to be of finite type. This requires every component of the collection $\{H_\alpha\}$ to be a finite dimensional algebra. Once each component has a finite dimensionality, he could easily construct their dual coalgebra. For any $\alpha \in \pi$, the $\alpha^{\text{th}}$ component of his quantum double $D(H)$, denoted by $D_\alpha(H)$ is a vector space, given as

$$D(H)_\alpha = H_{\alpha^{-1}} \otimes \big( \oplus_{\beta\in\mathcal{G}} H^*_\beta \big).$$

In our setting, we are working with finite dual cosections, of the dual cosheaf $\mathcal{A}^\circ$. Thus finite dimensionality is not required explicitly. As a result, our quantum double will be a quotient of the quantum double given by Zunino's construction. We define Drinfeld double of a crossed Hopf $\mathcal{G}$-coalgebra $\mathcal{A}$ as the sheaf $\widetilde{D(A)}$ such that:

$$D(A) := A \otimes_{\mathbb{K}} A^\circ$$

where $\mathcal{G} = \operatorname{Spec}(H)$ for a commutative $\mathbb{K}$-Hopf algebra $H$ and $A$ is an $H$-algebra with the action of $H$ on $A$ such that $^- : H \hookrightarrow A$ as a central Hopf $\mathbb{K}$-subalgebra

Finally, we conclude the thesis by discussing quantum double in the case when $\mathcal{G}$ is any general scheme (not necessarily affine). We propose that Drinfeld quantum double of a crossed Hopf $\mathcal{G}$-coalgebra $\mathcal{A}$ is a Hopf $\mathcal{G}$-coalgebra given by $D(\mathcal{A}) = \mathcal{A} \otimes \mathbb{K}\Gamma$, $\Gamma = \mathcal{A}^\circ(\mathcal{G})$, where $\Gamma$, the global cosection of the dual cosheaf is a Hopf algebra over the base field $\mathbb{K}$.

# Chapter 2

# Basic Results

In this chapter we shall give a brief account of some of the basic concepts and results that are essential this thesis.

Throughout, all the rings will be associative with an identity element.

## 2.1 Category Theory

Throughout the thesis, any general category shall be denoted as $\mathcal{C}$ such that Ob$\mathcal{C}$ is the collection of all the objects of $\mathcal{C}$ and $\mathrm{Hom}\,(X,Y)$ is the collection of all morphisms (arrows) from $X$ to $Y$ where $X, Y \in \mathrm{Ob}\,\mathcal{C}$. We shall write $\mathrm{End}\,(X)$ for the collection of all morphisms (arrows) from $X$ to $X$. We shall denote a monoidal category as $\mathcal{C} = (\mathcal{C}_0, \otimes, I, a, \lambda, \gamma)$ where $\mathcal{C}_0$ is a category, $\otimes : \mathcal{C}_0 \times \mathcal{C}_0 \to \mathcal{C}_0$ is a functor called the tensor product of $\mathcal{C}$, an object $I$ of $\mathcal{C}_0$ called the unit of $\mathcal{C}$ and three natural families of isomorphisms:

$$a = a_{X,Y,Z} \colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

called the *associator* of $\mathcal{C}$,

$$\lambda = \lambda_X \colon I \otimes X \to X \quad , \quad \gamma = \gamma_X \colon X \otimes I \to X$$

called the left and right *unitors* of $\mathcal{C}$ satisfy the coherence conditions (pentagon axiom for associativity and triangle axiom for unitors). $\mathcal{C}$ is *strict* when each $a, \lambda, \gamma$ is an identity arrow in $\mathcal{C}$. We shall denote a braided monoidal category as $\mathcal{C} = (\mathcal{C}_0, \otimes, I, a, \lambda, \gamma, \tau)$ where

$\mathcal{C}$ is a monoidal category together with a natural family of isomorphisms

$$\tau = \tau_{X,Y} : X \otimes Y \to Y \otimes X$$

called a *braiding* in $\mathcal{C}$ that is natural in both $X$ and $Y$, such that it satisfies the coherence conditions(the hexagonal axiom for the braiding). A *symmetry* is a braiding $\tau$ such that $\tau^2 = 1$ in $\mathcal{C}$ and in such a case we say $\mathcal{C} = (\mathcal{C}_0, \otimes, I, a, \lambda, \gamma, \tau)$ is a symmetric monoidal category.

In a monoidal category $\mathcal{C}$, a *monoid* (or monoid object) $(M, \mu, \eta)$ is an object $M$ together with two morphisms $\mu : M \otimes M \to M$ called multiplication and $\eta : I \to M$ called unit such that $\mu$ and $\eta$ satisfies the usual axioms of associativity and unity respectively. Dually, a *comonoid* in a monoidal category $\mathcal{C}$ is a monoid in the dual category $\mathcal{C}^{\mathrm{op}}$. Explicitly, a comonoid $(L, \Delta, \epsilon)$ is an object $L$ together with morphisms $\Delta : L \to L \otimes L$ called comultiplication and $\epsilon : L \to I$ called counit such that $\Delta$ and $\epsilon$ satisfies the usual axioms of coassociativity and counity respectively.

Suppose that the monoidal category $\mathcal{C}$ has a symmetry $\tau$. A monoid $M$ in $\mathcal{C}$ is *symmetric* when $\mu \circ \tau = \mu$. Given two monoids $(M, \mu, \eta)$ and $(M', \mu', \eta')$ in a monoidal category $\mathcal{C}$, a morphism $f : M \to M'$ is a morphism of monoids when $f$ is compatible with both $\mu$ and $\eta$. Explicitly, this would require $f$ to satisfy $f \circ \mu = \mu' \circ (f \otimes f)$ and $f \circ \eta = \eta'$.

In a monoidal category $(\mathcal{C}, \otimes, I, a, \lambda, \gamma)$, a *pair of dual* objects is a pair $(X, Y)$ of objects together with two morphisms $u_X : I \to X \otimes Y$ and $v_X : Y \otimes X \to I$ such that they satisfy

$$\gamma_X \circ (\mathrm{id}_X \otimes v_X) \circ a_{X,Y,X} \circ (u_X \otimes \mathrm{id}_X) \circ \lambda_X^{-1} = \mathrm{id}_X,$$

and

$$\lambda_Y \circ (v_X \otimes \mathrm{id}_Y) \circ a^{-1}_{\mathrm{id}_Y, X, Y} \circ (Y \otimes u_X) \circ \gamma_Y^{-1} = \mathrm{id}_Y.$$

In this situation, the object $Y$ is called a left dual of $X$, and $X$ is called a right dual of $Y$. Left duals are canonically isomorphic when they exist, as are right duals. When $\mathcal{C}$ is braided (or symmetric), every left dual is also a right dual, and vice versa. Further, a monoidal category $\mathcal{C}$ is called *rigid* if every object in $\mathcal{C}$ has right and left duals.

A parallel notion for duals is a non-degenerate pairing. A *pairing* of two objects $X$ and $Y$ in $\mathcal{C}$ is simply a morphism $v : X \otimes Y \to I$ in $\mathcal{C}$. Such a pairing (denoted by $v_X$) is *non-degenerate* in $X$ if there exists a morphism $u_X : I \to Y \otimes X$ called a copairing, such that the first equation above is satisfied.

Similarly, the pairing will be *non-degenerate* in $Y$ if there exists a morphism $u_Y : I \to Y \otimes X$ again called a copairing, such that the second equation above (with $u_X$ replaced by $u_Y$) is satisfied.

The important notion comes when this pairing is non-degenerate simultaneously in both $X$ and $Y$; and in that case we say $v$ to be simply non-degenerate.

**Proposition 2.1.1** *If a pairing is non-degenerate in both $X$ and $Y$, then the two copairings are the same.*

PROOF: Let us denote $\beta_X$ a copairing which makes $\beta$ nondegenerate in $X$, and let $\beta_Y$ be the copairing which makes $\beta$ nondegenerate in $Y$. So we have $(\beta \otimes id_X)(id_X \otimes \beta_X) = id_X$ and $(id_Y \otimes \beta)(\beta_Y \otimes id_Y) = id_Y$. Now consider the composite $\widehat{\beta}$ defined as

$$I \xrightarrow{\beta_X \otimes \beta_Y} X \otimes Y \otimes X \otimes Y \xrightarrow{id_X \otimes \beta \otimes id_Y} X \otimes Y.$$

We also can factor $\hat{\beta}$ like this :

$$I \xrightarrow{\beta_X} Y \otimes X \xrightarrow{\beta_Y \otimes id_Y \otimes id_X} (Y \otimes X) \otimes Y \otimes X \xrightarrow{a} Y \otimes (X \otimes Y) \otimes X \xrightarrow{id_Y \otimes \beta \otimes id_X} Y \otimes X.$$

Using the nondegeneracy in $Y$, this gives:

$$\begin{aligned}
\widehat{\beta} &= (id_Y \otimes \beta \otimes id_X)(\beta_Y \otimes id_Y \otimes id_X)\beta_X \\
&= \Big[(id_Y \otimes \beta)(\beta_Y \otimes id_Y) \otimes (id_X \otimes id_X)\Big]\beta_X \\
&= (id_Y \otimes id_X)\beta_X \\
&= \beta_X.
\end{aligned}$$

We factor $\widehat{\beta}$ as below :

$$I \xrightarrow{\beta_Y} Y \otimes X \xrightarrow{id_Y \otimes id_X \otimes \beta_X} Y \otimes X \otimes (Y \otimes X) \xrightarrow{a} Y \otimes (X \otimes Y) \otimes X \xrightarrow{id_Y \otimes \beta \otimes id_X} Y \otimes X.$$

Using the nondegeneracy in $X$, this gives:

$$\begin{aligned}
\widehat{\beta} &= (id_Y \otimes \beta \otimes id_X)(id_Y \otimes id_X \otimes \beta_X)\beta_Y \\
&= [id_Y \otimes (\beta \otimes id_X)(id_X \otimes \beta_X)]\beta_Y \\
&= (id_Y \otimes id_X)\beta_Y \\
&= \beta_Y.
\end{aligned}$$

□

Thus having a non-degenerate pairing for a pair of objects $(X, Y)$ in $\mathcal{C}$ is equivalent to say that $(X, Y)$ is a pair of dual objects in $\mathcal{C}$. Conversely, if any object $X$ in the monoidal category $\mathcal{C}$ has a dual $(X^*, u_X, v_X)$, then $v_X : X^* \otimes X \longrightarrow I$ forms a non-degenerate pairing with $u_X$ as its copairing. Note that one does not need to be in a rigid category for defining these notions. Simply a monoidal category will do.

Suppose $(X, X^*)$ is a pair of dual objects of a symmetric monoidal category $\mathcal{C}$. Then for any morphism $f : X \to X$, we define its *trace* $tr(f)$ to be the following composition of morphisms:

$$I \to X \otimes X^* \xrightarrow{f \otimes 1} X \otimes X^* \xrightarrow{\tau} X^* \otimes X \to I.$$

Thus, we can also think of it as a morphism $tr_X : \mathrm{Hom}\,(X) \to \mathrm{Hom}\,(I)$ given by the above composition. The definition can be immediately generalised. Namely, let now $P$ and $Q$ are other two objects in $\mathcal{C}$. Then for any $f : P \otimes X \to Q \otimes X$ we define the trace morphism $tr_X : \mathrm{Hom}\,(P \otimes X, Q \otimes X) \to \mathrm{Hom}\,(P, Q)$ as follows:

$$P \cong P \otimes I \to P \otimes X \otimes X^* \xrightarrow{f \otimes 1} Q \otimes X \otimes X^* \xrightarrow{1 \otimes \tau} Q \otimes X^* \otimes X \to Q \otimes I \cong Q.$$

A *preadditive* category $\mathcal{C}$ is a category in which each hom-set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an additive abelian group for which composition is bilinear. For morphisms $f, f' : X \to Y$

and $g, g^{'} : Y \to Z$,

$$(g + g^{'}) \circ (f + f^{'}) = g \circ f + g \circ f^{'} + g^{'} \circ f + g^{'} \circ f^{'}.$$

Thus, *Ab*, *R-Mod*, *Mod-R* are preadditive categories. A preadditive category is called *additive* if the following conditions are satisfied:

(i) There is a zero object $0 \in \mathrm{Ob}\,\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(0, X) = \mathrm{Hom}_{\mathcal{C}}(X, 0) = 0$ for all $X \in \mathrm{Ob}\,\mathcal{C}$.

(ii) Every finite set of objects has a product. This means that we can form finite direct products.

If $\mathcal{C}$ and $\mathcal{D}$ are preadditive categories, an *additive functor* $T : \mathcal{C} \to \mathcal{D}$ is a functor from $\mathcal{C}$ to $\mathcal{D}$ with

$$T(f + f^{'}) = Tf + Tf^{'}.$$

for any parallel pair of arrows $f, f^{'} : X \to Y$ in $\mathcal{C}$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $T : \mathcal{C} \to \mathcal{D}$ and $S : \mathcal{D} \to \mathcal{C}$ be covariant functors. $T$ is said to be a *left adjoint functor* to $S$ (equivalently, $S$ is a *right adjoint functor* to $T$) if there is a natural isomorphism :

$$\nu : \mathrm{Hom}_{\mathcal{D}}(T(-), -) \stackrel{\cdot}{\to} \mathrm{Hom}_{\mathcal{C}}(-, S(-)).$$

Here the functor $\mathrm{Hom}_{\mathcal{D}}(T(-), -)$ is a bifunctor $\mathcal{C} \times \mathcal{D} \to \mathbf{Set}$ which is contravariant in the first variable, is covariant in the second variable, and sends an object $(C, D)$ to $\mathrm{Hom}_{\mathcal{D}}(T(C), D)$. The functor $\mathrm{Hom}_{\mathcal{C}}(-, S(-))$ is defined analogously. Essentially, it says that for every object $C$ in $\mathcal{C}$ and every object $D$ in $\mathcal{D}$ there is a function

$$\nu_{C,D} : \mathrm{Hom}_{\mathcal{D}}(T(C), D) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(C, S(D))$$

which is a natural bijection of hom-sets. Naturality means that if $f : C^{'} \to C$ is a morphism in $\mathcal{C}$ and $g : D \to D^{'}$ is a morphism in $\mathcal{D}$, then the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}\big(T(C), D\big) \arrow[r, "\nu_{C,D}"] \arrow[d, "(Tf{,}g)"'] & \mathrm{Hom}_{\mathcal{C}}\big(C, S(D)\big) \arrow[d, "(f{,}Sg)"] \\
\mathrm{Hom}_{\mathcal{D}}\big(T(C^{'}), D^{'}\big) \arrow[r, "\nu_{C^{'},D^{'}}"] & \mathrm{Hom}_{\mathcal{C}}\big(C^{'}, S(D^{'})\big).
\end{tikzcd}$$

commutes. If we pick any $h : T(C) \to D$, then we have the equation

$$Sg \circ \nu_{C,D}(h) \circ f = \nu_{C^{'},D^{'}}(g \circ h \circ Tf).$$

If $T : \mathcal{C} \to \mathcal{D}$ is a left adjoint of $S : \mathcal{D} \to \mathcal{C}$, then we say that the ordered pair $(T, S)$ is an *adjoint pair*, and the ordered triple $(T, S, \nu)$ an *adjunction* from $\mathcal{C}$ to $\mathcal{D}$, written as

$$(T, S, \nu) : \mathcal{C} \to \mathcal{D},$$

where $\nu$ is the natural equivalence defined above. An adjoint to any functor is unique up to natural isomorphism.

Let $(\mathcal{C}, \otimes, I_{\mathcal{C}})$ and $(\mathcal{D}, \otimes, I_{\mathcal{D}})$ be monoidal categories. A *lax monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ consists of a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ together with a natural transformation

$$\phi_{A,B} : \mathcal{F}A \otimes \mathcal{F}B \to \mathcal{F}(A \otimes B)$$

and a morphism

$$\phi : I_D \to \mathcal{F}I_C,$$

called the coherence maps or structure morphisms, which are such that for any three objects $A$, $B$ and $C$ of $\mathcal{C}$ the diagrams

$$\begin{tikzcd}
(\mathcal{F}A \otimes \mathcal{F}B) \otimes \mathcal{F}C \arrow[r, "a_{\mathcal{D}}"] \arrow[d, "\phi_{A,B}\otimes 1"'] & \mathcal{F}A \otimes (\mathcal{F}B \otimes \mathcal{F}C) \arrow[d, "1\otimes\phi_{B,C}"] \\
\mathcal{F}(A \otimes B) \otimes \mathcal{F}C \arrow[d, "\phi_{A\otimes B,C}"'] & \mathcal{F}A \otimes \mathcal{F}(B \otimes C) \arrow[d, "\phi_{A,B\otimes C}"] \\
\mathcal{F}\big((A \otimes B) \otimes C\big) \arrow[r, "\mathcal{F}a_{\mathcal{C}}"] & \mathcal{F}\big(A \otimes (B \otimes C)\big)
\end{tikzcd} \tag{2.1}$$

$$\begin{tikzcd}
\mathcal{F}A \otimes I_{\mathcal{D}} \arrow[r, "1\otimes\phi"] \arrow[d, "\rho_{\mathcal{D}}"'] & \mathcal{F}A \otimes \mathcal{F}I_{\mathcal{C}} \arrow[d, "\phi_{A,I_{\mathcal{C}}}"] \\
\mathcal{F}A & \mathcal{F}(A \otimes I_{\mathcal{C}}). \arrow[l, "\mathcal{F}\rho_{\mathcal{C}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I_{\mathcal{D}} \otimes \mathcal{F}B \arrow[r, "\phi\otimes 1"] \arrow[d, "\lambda_{\mathcal{D}}"'] & \mathcal{F}I_{\mathcal{C}} \otimes \mathcal{F}B \arrow[d, "\phi_{\mathcal{C},B}"] \\
\mathcal{F}B & \mathcal{F}(I_{\mathcal{C}} \otimes B). \arrow[l, "\mathcal{F}\lambda_{\mathcal{C}}"']
\end{tikzcd} \tag{2.2}$$

commute in the category $\mathcal{D}$. Above, the various natural transformations denoted using $\alpha$, $\rho$, $\lambda$ are parts of the monoidal structure on $\mathcal{C}$ and $\mathcal{D}$. A *monoidal functor* is a lax monoidal functor whose coherence maps $\phi_{A,B}$, $\phi$ are isomorphisms, and a *strict monoidal functor* is one whose coherence maps are identities.

Now suppose that the monoidal categories $\mathcal{C}$ and $\mathcal{D}$ are symmetric with braidings $c_{\mathcal{C}}$ and $c_{\mathcal{D}}$ respectively. Then the monoidal functor $\mathcal{F}$ is symmetric when the diagram

$$\begin{tikzcd}
\mathcal{F}A \otimes \mathcal{F}B \arrow[r, "c_{\mathcal{D}}"] \arrow[d, "\phi_{A,B}"'] & \mathcal{F}B \otimes \mathcal{F}A \arrow[d, "\phi_{B,A}"] \\
\mathcal{F}(A \otimes B) \arrow[r, "\mathcal{F}(c_{\mathcal{C}})"'] & \mathcal{F}(B \otimes A).
\end{tikzcd} \tag{2.3}$$

commutes for any objects $A$ and $B$ of $\mathcal{C}$.

Suppose $\mathcal{D}$ is a full subcategory of the monoidal category $\mathcal{C}$. Then there is an *inclusion functor*

$$\mathcal{G} : \mathcal{D} \hookrightarrow \mathcal{C}$$

that sends each object of $\mathcal{D}$ to itself (in $\mathcal{C}$), and each morphism of $\mathcal{D}$ to itself (in $\mathcal{C}$). This functor is a faithful functor. In addition, if $\mathcal{D}$ is a *monoidal subcategory* of $\mathcal{C}$, then $\mathcal{D}$ is closed under the tensor product of objects and morphisms and contains the identity object of $\mathcal{C}$. Thus $I_{\mathcal{D}} = I_{\mathcal{C}} = I$, say. Let

$$\mathcal{F} : \mathcal{C} \to \mathcal{D}$$

be a functor such that there is a natural isomorphism $\theta : 1 \to \mathcal{G}\mathcal{F}$, $\theta_A : A \mapsto \mathcal{F}A$. Then, $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories. We want to investigate whether the equivalence of monoidal categories a monoidal equivalence. The answer is negative in general and the equivalent monoidal categories $\mathcal{C}$ and $\mathcal{D}$ need not possess monoidal equivalence.

**Proposition 2.1.2** *The equivalence $\mathcal{F}$ is a monoidal equivalence if and only if the natural isomorphism $\theta$ (as described above) satisfies the following three commutative squares*

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "(\theta_A \otimes \theta_B) \otimes \theta_C"] \arrow[d, "a_{\mathcal{C}}"'] & (FA \otimes FB) \otimes FC \arrow[d, "a_{\mathcal{D}}"] \\
A \otimes (B \otimes C) \arrow[r, "\theta_A \otimes (\theta_B \otimes \theta_C)"] & FA \otimes (FB \otimes FC)
\end{tikzcd} \tag{2.4}$$

$$\begin{tikzcd}
\mathcal{F}A \otimes I \arrow[r, "\theta_A^{-1} \otimes 1"] \arrow[d, "\rho_{\mathcal{D}}"'] & A \otimes I \arrow[d, "\rho_{\mathcal{C}}"] \\
\mathcal{F}A & A \arrow[l, "\theta_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes \mathcal{F}A \arrow[r, "1 \otimes \theta_A^{-1}"] \arrow[d, "\lambda_{\mathcal{D}}"'] & I \otimes A \arrow[d, "\lambda_{\mathcal{C}}"] \\
\mathcal{F}A & A \arrow[l, "\theta_A"']
\end{tikzcd} \tag{2.5}$$

PROOF: The commutativity of the diagram (2.1) for the coherence maps is equivalent to (after all the cancellations) the commutativity of the diagram (2.4). Similarly, after all the cancellations the commutativity of the diagram in (2.2) becomes equivalent to the commutativity of the diagrams in (2.5) and respectively. Thus, it is a monoidal equivalence. □

Now suppose that the monoidal categories $\mathcal{C}$ and $\mathcal{D}$ are symmetric with braidings $c_{\mathcal{C}}$ and $c_{\mathcal{D}}$ respectively. Then equivalence $\mathcal{F}$ is a monoidal functor by Proposition (2.1.2). The monoidal equivalence $\mathcal{F}$ becomes symmteric if we have:

**Corollary 2.1.3** *The monoidal equivalence $\mathcal{F}$ is a symmetric monoidal equivalence if and only if the natural isomorphism $\theta$ (as described above) is such that the diagram*

$$\begin{tikzcd} \mathcal{F}A \otimes \mathcal{F}B \arrow[r, "c_{\mathcal{D}}"] \arrow[d, "\theta_A^{-1}\otimes\theta_B^{-1}"'] & \mathcal{F}B \otimes \mathcal{F}A \arrow[d, "\theta_B^{-1}\otimes\theta_A^{-1}"] \\ A \otimes B \arrow[r, "c_{\mathcal{C}}"'] & B \otimes A. \end{tikzcd} \tag{2.6}$$

*commutes. Here, $c_{\mathcal{C}}$ and $c_{\mathcal{D}}$ are the braidings of the categories $\mathcal{C}$ and $\mathcal{D}$ respectively.*

Proof: The commutativity of the diagram (2.3) for the braidings is equivalent to (after all the cancellations) the commutativity of the diagram (2.6). □

A covariant functor $T : \mathcal{C} \to \mathcal{D}$ is called a *Frobenius functor* if there exists a functor $S : \mathcal{D} \to \mathcal{C}$ which is a left as well as a right adjoint functor to $T$, and we say that the pair $(T, S)$ is Frobenius for $\mathcal{C}$ and $\mathcal{D}$. This notion is symmetric in $T$ and $S$, that is, if $(T, S)$ is a Frobenius pair, then so is the pair $(S, T)$. For details see [CMZ02].

A *monad* in a category $\mathcal{C}$ is a triple $T = (T, \mu, \eta)$, where $T : \mathcal{C} \to \mathcal{C}$ is a functor with natural transformations $\mu : TT \to T$ and $\eta : I_{\mathcal{C}} \to T$ satisfying associativity and unitary conditions. A morphism of monads $(T, \mu, \eta) \to (T', \mu', \eta')$ is a natural transformation $\varphi : T \to T'$ such that $\mu' \circ \varphi T' \circ T\varphi = \varphi \circ \mu$ and $\varphi \circ \eta = \eta'$.

Given a monad $T =< T, \mu, \eta >$ in $\mathcal{C}$, a $T$-*algebra* (or a $T$-module) $\langle A, \rho_A \rangle$ is a pair consisting of an object $A \in \mathrm{Ob}\,\mathcal{C}$ and a morphism $\rho_A : TA \to A$ in $\mathcal{C}$ (called the structure map of the algebra) which is associative and unital. A morphism $f : \langle A, \rho_A \rangle \to \langle A', \rho'_A \rangle$ of $T$-algebras is a morphism $f : A \to A'$ in $\mathcal{C}$ which preserves the structure maps.

**Theorem 2.1.4** *[ML98, p. 136] (Every monad is defined by its $T$-algebras.) If $(T, \mu, \eta)$ is a monad in $\mathcal{C}$, then the set of all $T$-algebras and their morphisms form a category $\mathcal{C}^T$, called Eilenberg-Moore category. There is an adjunction*

$$\langle F^T, G^T; \eta^T, \epsilon^T \rangle : \mathcal{C} \to \mathcal{C}^T$$

*in which the functors $G^T$ and $F^T$ are given by the respective assignments*

$$\begin{tikzcd} G^T: & \langle A, \rho_A \rangle \arrow[r, mapsto] \arrow[d, "f"] & A \arrow[d, "f"] \\ & \langle A', \rho_{A'} \rangle \arrow[r, mapsto] & A'. \end{tikzcd} \qquad \begin{tikzcd} F^T: & A \arrow[r, mapsto] \arrow[d, "f"] & \langle TA, \mu_A \rangle \arrow[d, "Tf"] \\ & A' \arrow[r, mapsto] & \langle TA', \mu_{A'} \rangle. \end{tikzcd}$$

*while $\eta^T = \eta$ and $\epsilon^T \langle A, \rho_A \rangle = \rho_A$ for each $T$-algebra $\langle A, \rho_A \rangle$. the monad defined in $\mathcal{C}$ by this adjunction is the given monad $(T, \mu, \eta)$.*

Dually, one can define *comonads* on $\mathcal{C}$. A comonad on $\mathcal{C}$ is a triple $W = (W, \delta, \epsilon)$, where $W : \mathcal{C} \to \mathcal{C}$ is a functor with natural transformations $\delta : \mathcal{G} \to \mathcal{G}\mathcal{G}$ and $\epsilon : \mathcal{G} \to I_{\mathcal{C}}$ satisfying coassociativity and counitary conditions. A morphism of comonads is a natural transformation that is compatible with the coproduct and counit.

Given a monad $W =< W, \delta, \epsilon >$ in $\mathcal{C}$, a $W$-*coalgebra* (or a $W$-comodule) $\langle C, \rho^C \rangle$ is a pair consisting of an object $C \in \mathrm{Ob}\, \mathcal{C}$ and a morphism $\rho^C : C \to TC$ in $\mathcal{C}$ (called the structure map of the coalgebra) which is coassociative and counital. A morphism $f : \langle A, \rho_A \rangle \to \langle A', \rho'_A \rangle$ of $T$-algebras is a morphism $f : A \to A'$ in $\mathcal{C}$ which preserves the structure maps.

The Eilenberg-Moore category of $W$-coalgebras is denoted by $\mathcal{C}^{\mathcal{G}}$. For any $A \in \mathrm{Ob}\, \mathcal{C}$, $\mathcal{G}A$ is a $W$-comodule giving the cofree functor

$$\phi^W : \mathcal{C} \to \mathcal{C}^W \ , \ A \mapsto (WA, \delta A)$$

which is right adjoint to the forgetful functor $\mathrm{For}^W : \mathcal{C}^W \to \mathcal{C}$ by the isomorphism

$$\mathrm{Hom}_{\mathcal{C}^W}(B, \phi^W A) \to \mathrm{Hom}_{\mathcal{C}}(\mathrm{For}^W B, A), \ f \mapsto \epsilon A \circ f.$$

A *free functor* is a left adjoint to a forgetful functor. (This is a very informal way of defining because the concept of forgetful functor is informal; any functor might be viewed as forgetful, so any left adjoint might be viewed as free, while in practice only some are.)

**Examples.**

(i) the free monoid functor Set$\to$ Mon.

(ii) the free module functor Set$\to$ $K$-Mod for a ring $K$.

(iii) the free group functor Set$\to$ Grp.

A very important example (concept) forming a free functor is the left adjoint $\mathcal{C} \to \mathcal{C}^{\mathcal{F}}$, where $\mathcal{F}$ is a monad on the category $\mathcal{C}$ and $\mathcal{C}^{\mathcal{F}}$ is its Eilenberg-Moore category. This includes all of the three examples above.

## 2.2 Crossed modules

Crossed modules were first introduced by J. H. C. Whitehead as a tool for homotopy theory. They also occur naturally in many other situations (see examples below).

**Definition 2.2.1** *A crossed module $\mathcal{H} = (H, \pi, t, \varphi)$ consists of groups $H$, $\pi$ together with a group homomorphism $t : H \to \pi$ and a left action $\varphi : \pi \times H \to H$ on $H$, written as ${}^{\alpha}h := \varphi(\alpha, h)$, satisfying the conditions*

**CM1** $t({}^{\alpha}h) = \alpha t(h)\alpha^{-1}$

**CM2** ${}^{t(h)}h' = hh'h^{-1}$.

When the action is unambiguous, we may write $\mathcal{H}$ as a triple $(H, \pi, t)$. The two crossed module axioms also have names, which are inconsistently applied. **CM1** is sometimes known as *equivariance*; **CM2** is called the *Peiffer identity*. A structure with the same data as a crossed module and satisfying the equivariance condition but not the Peiffer identity is called a *precrossed module*.

**Examples.** Certain generic situations give rise to crossed modules. Some are detailed here.

(i) Suppose $N \lhd G$ is a normal subgroup. Then $G$ acts on $N$ by conjugation; this action and the inclusion $i : N \hookrightarrow G$ form a conjugation crossed module, $(N, G, i)$.

(ii) If $M$ is a $G$-module, there is a well-defined $G$-action on $M$. This together with the zero homomorphism : $M \to G$ (sending everything in $M$ to the identity in $G$) yields a $G$-module crossed module, $(M, G, 0)$.

(iii) Let $G$ be any group and $\mathrm{Aut}(G)$ its group of automorphisms. There is an obvious action of $\mathrm{Aut}(G)$ on $G$, and a homomorphism $\phi : G \to \mathrm{Aut}(G)$ sending each $g \in G$ to the inner automorphism of conjugation by $g$. These together form an automorphism crossed module, $(G, \mathrm{Aut}(G), \phi)$.

(iv) Any group $G$ may be thought of as a crossed module in two ways. Since $G$ always has the two normal subgroups $\{1\}$ and $G$, we can form the conjugation crossed modules $\{1\} \hookrightarrow G$ and $\mathrm{id} : G \to G$. Note that the homomorphism $G \to \{1\}$ with the trivial action forms a crossed module whenever $G$ is abelian, otherwise the Peiffer identity fails and the result is a precrossed module.

## 2.3 Cohomology of groups

Let $G$ be a multiplicative group. Suppose $M$ is an abelian group. Then $M$ can be regarded as a trivial $G$-module (via the action $x\mu = \mu$ for $x \in G$, $\mu \in M$) and the cohomology groups $H^n(G, M)$ can be constructed.

For $n \geq 0$, let $C^n(G, M)$ be the group of all functions from $G^n$ to $M$. This is an abelian group; its elements are called the (inhomogeneous) $n$-cochains. The coboundary homomorphisms

$$d^n : C^n(G, M) \to C^{n+1}(G, M)$$

are defined as

$$\begin{aligned}(d^n\varphi)(g_1, \ldots, g_{n+1}) \quad = \quad & \varphi(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi\big(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}\big) \\ & (-1)^{n+1}\varphi\big(g_1, \ldots, g_n\big).\end{aligned}$$

The crucial thing to check here is

$$d^{n+1} \circ d^n = 0.$$

Thus we have a cochain complex and we can compute cohomology. For $n \geq 0$, define the group of n-cocycles as: $Z^n(G,M) = \ker(d^n)$ and the group of n-coboundaries as

$$B^n(G,M) = \begin{cases} 0 & ; n = 0 \\ \mathrm{im}(d^{n-1}) & ; n \geq 1 \end{cases}$$

and finally the cohomology group of $G$ with coefficients in $M$ is defined as

$$H^n(G,M) = Z^n(G,M)/B^n(G,M).$$

Let $\theta$ be a 2-cocycle of $G$. This implies $\theta \in Z^2(G,M)$. Then $d^2(\theta) = 0$. Thus for any $f, g, h \in G$ using the definition of the differential $d^2$, we get

$$\theta(f, gh) + \theta(g, h) = \theta(f, g) + \theta(fg, h) \tag{2.7}$$

A 2-cocycle $\theta$ of $G$ is called *normalised* if

$$\theta(f, 1) = \theta(1, g) = 0. \tag{2.8}$$

Further, a 3-cochain $\sigma$ of $G$ is a coboundary of $\theta$ implies that $d\theta = \sigma$. Thus we have:

$$\sigma(f, g, h) + \theta(f, gh) + \theta(g, h) = \theta(f, g) + \theta(fg, h) \tag{2.9}$$

Suppose $\sigma$ is a 3-cocycle of $G$. This implies $\sigma \in Z^3(G,M)$. Then $d^3(\sigma) = 0$. Thus for any $a, b, c, d \in G$ using the definition of the differential, we get

$$\sigma(a, b, c) + \sigma(d, ab, c) + \sigma(d, a, b) = \sigma(d, a, bc) + \sigma(da, b, c)$$

A 3-cocycle $\sigma$ is called *normalised* if

$$\sigma(f, 1, h) = 0.$$

Suppose $G = \mathbb{Z}$, then we have

$$\begin{aligned} H^0(G, M) &= M \\ H^i(G, M) &= 0 \text{ for } i \geq 1. \end{aligned}$$

In particular, $H^3(\mathbb{Z}, M) = 1$ since $\mathbb{Z}$ is a projective $\mathbb{Z}$-module, where as for any cyclic group of order $n$, $H^3(\mathbb{Z}_n, M) = {}_nM$ where ${}_nM = \{\mu \in M \big| n\mu = 0\}$, and a standard 3-cocycle that gives the cohomology class of $\mu \in {}_nM$ is given by the standard formula:

$$\sigma(x, y, z) = \begin{cases} 0 & ; y + z < n \\ xn\mu & ; y + z \geq n \end{cases}$$

for some $\mu \in {}_nM$.

## 2.4 Duality and Pairings

Suppose $\mathbb{S}$ is a commutative ring with unity. Let $A$ be an $\mathbb{S}$-algebra. We say that an ideal $I$ of $A$ is *finite coprojective* if $A/I$ is finitely generated projective $\mathbb{S}$-module. We define a *finite dual* $A^\circ$ of $A$ to be the $\mathbb{S}$-algebra given by

$$A^\circ = \{f \in A^* \big| f(I) = 0 \ ; \ I \text{ is finite coprojective.}\} \tag{2.10}$$

Now consider two Hopf algebras $A$ and $B$ (over $\mathbb{K}$). A *Hopf Pairing* between them is a bilinear pairing $\sigma : A \times B \to \mathbb{K}$ such that, for all $a, a^{'} \in A$ and $b, b^{'} \in B$,

$$\sigma(a, bb^{'}) = \sigma(a_{(1)}, b)\sigma(a_{(2)}, b^{'}), \tag{2.11}$$

$$\sigma(aa^{'}, b) = \sigma(a, b_{(2)})\sigma(a^{'}, b_{(1)}), \tag{2.12}$$

$$\sigma(a, 1) = \epsilon(a) \quad \text{and} \quad \sigma(1, b) = \epsilon(b) \tag{2.13}$$

Note that such a pairing always verifies that, for any $a \in A$ and $b \in B$,

$$\sigma(S(a), S(b)) = \sigma(a, b), \tag{2.14}$$

since both $\sigma$ and $\sigma(S \otimes S)$ are the inverse of $\sigma(id \otimes S)$ in the algebra $\mathrm{Hom}_{\mathbb{K}}(A \otimes B, \mathbb{K})$ endowed with the convolution product.

## 2.5 Schemes

Let $\mathbb{K}$ be a commutative ring. An *affine scheme* $\mathcal{X}$ over $\mathbb{K}$ is a locally ringed space $(X, \mathcal{O}_X)$ such that its base space $X$ is isomorphic to $\operatorname{Spec} R$ for some commutative $\mathbb{K}$-algebra $R$. If $x \in R$, define the basic open set $X_x = \{\mathfrak{p} \in \operatorname{Spec} R : x \notin \mathfrak{p}\}$. It is the locus where $x$ does not vanish. Then $\{X_x\}_{x \in R}$ forms a prebasis of $X$ and equips $X$ with a Zariski topology.

Consider the multiplicative subset $S = R - \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of $R$. The localisation $S^{-1}R$ is denoted by $R_{\mathfrak{p}}$. For a non zero $x$ in $R$, the localisation of the multiplicative subset $\{1, x, x^2, \cdots\}$ denoted by $R_x$ is just the ring obtained by inverting powers of $x$. Note if $x$ is nilpotent, the localisation is the zero ring. (Note : If $(x)$ is a prime ideal, then $R_x \neq R_{(x)}$ .) Here is an example: $R = \mathbb{C}[X, Y]$, $x = X$. In this case $R_X = \mathbb{C}[X, X^{-1}, Y]$. On the other hand every nonzero element of $R_{(X)} = \{R \backslash (X)\}^{-1} R$ has a unique representation $X^n u$, where $u$ is a fraction, such that $X$ is coprime to both the numerator and the denominator of $u$. We can call this ring the local ring of the affine plane at the line $X = 0$ i.e. $R_{(X)}$ is the ring of rational functions on $\mathbb{C}^2$ which do not have a pole along the hyperplane $X = 0$.

A *scheme* $\mathcal{X}$ over $\mathbb{K}$ is a locally ringed space $(X, \mathcal{O}_X)$ admitting an open covering $\{U_i\}_{i \in I}$, such that for each $i$, $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic as locally ringed space to an affine scheme over $\mathbb{K}$ . Points of $X$ are referred to as topological points of $X$ or $X^{top}$. A morphism of schemes is a morphism of locally ringed spaces. An isomorphism will be a morphism with two-sided inverse. Indeed, schemes form a category, Sch. Morphisms from schemes to affine schemes are completely understood in terms of ring homomorphisms by the following contravariant adjoint pair: For any scheme $\mathcal{X}$ and a commutative $\mathbb{K}$-algebra $R$ we have a natural equivalence

$$\mathrm{Mor}_{\mathrm{Sch}}(\mathcal{X}, \operatorname{Spec} R) \cong \mathrm{Mor}_{\mathbb{K}-\mathrm{Alg}}\big(R, \mathcal{O}_X(X)\big),$$

where $\mathbb{K} - \mathrm{Alg}$ is the category of commutative $\mathbb{K}$-algebras. Since $\mathbb{K}$ is an initial object in the category $\mathbb{K} - \mathrm{Alg}$, the category of schemes has $\mathrm{Spec}\,(\mathbb{K})$ as a final object.

The category of schemes has finite products. By definition, the product of $\mathcal{X}$ and $\mathcal{Y}$ in the category is the unique object satisfying the universal property that if $\mathcal{Z}$ is any object in Schemes with maps to $\mathcal{X}$ and $\mathcal{Y}$, then both maps factor uniquely through a map from $\mathcal{Z} \to \mathcal{X} \times \mathcal{Y}$. A generalisation of this notion is that of the fiber product. Let now $\mathcal{X}$ and $\mathcal{Y}$ be be schemes over a base scheme $\mathcal{S}$. This means both $\mathcal{X}$ and $\mathcal{Y}$ are equipped with some specified scheme morphisms to the base scheme $\mathcal{S}$. Then any object $\mathcal{Z}$ with maps to both $\mathcal{X}$ and $\mathcal{Y}$ commuting with these specified maps factors uniquely through the fiber product of $\mathcal{X}$ and $\mathcal{Y}$, denoted as $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$.

Before we construct the fiber product for arbitrary schemes, let us take a look at the case that $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{S}$ are all affine. We know that the category of affine schemes is equivalent to the category of commutative rings, with all the arrows reversed. So, in order to find a product of affine schemes, it suffices to find a coproduct of commutative rings. In other words, a ring $R$ with the universal property described by the diagram below:

$$\begin{tikzcd}
& & A \arrow[dll] \arrow[dl] & \\
Z & R \arrow[l, dotted, "\exists!"'] & & C \arrow[ul] \arrow[dl] \\
& & B \arrow[ull] \arrow[ul] &
\end{tikzcd}$$

where $\mathcal{X} = \mathrm{Spec}\, A$, $\mathcal{Y} = \mathrm{Spec}\, B$, $\mathcal{S} = \mathrm{Spec}\, C$, $\mathcal{Z} = \mathrm{Spec}\, Z$. As we can see, this is exactly the universal property of the tensor product. It follows that $\mathrm{Spec}\, A \times_{\mathrm{Spec}\, C} \mathrm{Spec}\, B = \mathrm{Spec}\,(A \otimes_C B)$.

But one has to be careful: the underlying topological space of the product scheme of $\mathcal{X} = (X, \mathcal{O}_{\mathcal{X}})$ and $\mathcal{Y} = (Y, \mathcal{O}_{\mathcal{Y}})$ is normally not equal to the product of the topological spaces $X$ and $Y$. In fact, the underlying topological space of the product scheme often has more points than the product of the underlying topological spaces. Let us look at some examples.

**Examples.**

(i) If $\mathbb{K}$ is the field with nine elements, then $\operatorname{Spec}\mathbb{K} \times \operatorname{Spec}\mathbb{K} \approx \operatorname{Spec}(\mathbb{K} \otimes_{\mathbb{Z}} \mathbb{K}) \approx \operatorname{Spec}(\mathbb{K} \otimes_{\mathbb{Z}_3} \mathbb{K}) \approx \operatorname{Spec}(\mathbb{K} \times \mathbb{K})$, a set with two elements, though $\operatorname{Spec}\mathbb{K}$ has only a single element.

(ii) If $\mathbb{K}$ is algebraically closed, then as a set for closed points, $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$. Observe the following: $\mathbb{A}^1 \times_{\operatorname{Spec}\mathbb{K}} \mathbb{A}^1 = \operatorname{Spec}\mathbb{K}[x] \times_{\operatorname{Spec}\mathbb{K}} \operatorname{Spec}\mathbb{K}[y] = \operatorname{Spec}(\mathbb{K}[x] \otimes_{\mathbb{K}} \mathbb{K}[y]) = \operatorname{Spec}\mathbb{K}[x,y] = \mathbb{A}^2$. Thus the product of the closed points of $\mathbb{A}^1$ with itself is $\mathbb{A}^2$ considered as a set. But note that the underlying topological space of $\mathbb{A}^2$ is not the product topology of the underlying topological space of $\mathbb{A}^1$ with itself. The product topology on $\mathbb{A}^2$ is the finite complement topology but the topology on $\mathbb{A}^2$ is different.

(iii) Another important example of a fiber product is the case where $f : X \to S$ is a morphism, and $\alpha$ is a point in $S$. Then consider $X \times_S \alpha$. If $X$ and $S$ were sets, this would be the set of points of $X$ whose image in $S$ is $\alpha$. In other words, the preimage or fiber of $\alpha$. From this perspective, we can think of $f : X \to S$ as a family of schemes parametrised by $S$, where the member of the family corresponding to a point $\alpha$ in $S$ is the fiber of $f$ over $\alpha$.

So far we have looked at the fiber product of affine schemes. We ought to construct the fiber product of arbitrary schemes. Since we have already done the affine case, we just need to patch the local products together to get some sort of global product. Moreover, notice that since the fiber product satisfies a universal property, it is unique up to unique isomorphism. This uniqueness is important for proving the following result also discussed by Hartshorne [Har77] :

**Theorem 2.5.1** *For any $\mathcal{X}$, $\mathcal{Y}$ schemes over $\mathcal{S}$, the fiber product $\mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ exists.*

Suppose $\mathcal{X}$ is a scheme, and $R$ is a commutative $\mathbb{K}$-algebra. If for all non-empty open sets $U$, the sections $\Gamma(U, \mathcal{O}_X)$ are $R$-algebras and the restrictions are $R$-algebra maps, then we say that $\mathcal{X}$ is an $R-$scheme, or a *scheme over* $R$. We also have an equivalence of the category of $\mathbb{K}-$schemes with a full sub category of the category of functors from commutative $\mathbb{K}$-algebras to *Sets*. So we think of an affine $\mathbb{K}-$scheme $\mathcal{X} = (X, \mathcal{O}_X)$ where $X = \operatorname{Spec} R$ as a functor such that for any $\mathbb{K}$-algebra $A$, $X(A) := \operatorname{Hom}_{\mathbb{K}}(R, A)$ as a set or equivalently $X(A) := \mathrm{Mor}_{\mathrm{Sch}}(\operatorname{Spec} A, X)$. Thus points of $X$ are now referred to as points of $X$ corresponding to $A$, or $A-$points. If $A$ is not specified, we simply call them algebraic points of $X$. Note that points must be $\mathbb{K}$-linear.

If $X = \bigcup_i (U_i, \mathcal{O}_{U_i})$, such that for each $i$, $U_i \cong \operatorname{Spec} R_i$ and $U_i \cap U_j = \operatorname{Spec} R_{ij}$ with $R_i \xrightarrow{\alpha_{ij}} R_{ij}$. Define $X(A) = \bigcup_i U_i(A)/\sim$ where $(x_i : R_i \to A) \sim (x_j : R_j \to A)$ if and only if either $i = j$ or $x_i|_{R_{ij}} = x_j|_{R_{ij}}$.

A scheme $\mathcal{X} = (X, \mathcal{O}_X)$ is said to be *locally of finite type* over $\mathbb{K}$ if corresponding to an open covering $\{U_i\}_{i\in I}$ of $X$, each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic as locally ringed space to $(\operatorname{Spec}(R_i), \mathcal{O}_{R_i})$, where $R_i$'s are finitely generated algebras over $\mathbb{K}$. A scheme is said to be *of finite type* over $\mathbb{K}$ if it is a scheme locally of finite type over $\mathbb{K}$, and the open cover is taken as finite. A scheme of finite type over $\mathbb{K}$ is affine if the open cover can consist of precisely one open set. This means $(X, \mathcal{O}_X)$ is isomorphic as a ringed space over $\mathbb{K}$, to $(\operatorname{Spec} R, \mathcal{O}_R)$ for some finitely generated $\mathbb{K}$-algebra $R$. A scheme $(X, \mathcal{O}_X)$ is said to be *reduced* if for every open subset $U \in X$, the ring $\mathcal{O}_X(U)$ has no non-zero nilpotent elements.

Before we describe group schemes over $\mathbb{K}$, we discuss diagonal and structure morphisms of a scheme $(X, \mathcal{O}_X)$ over $\mathbb{K}$.

For each affine open subset $U = \operatorname{Spec}(R)$ of a scheme $\mathcal{X} = (X, \mathcal{O}_X)$, let $\phi_R$ be the canonical injection of $\mathbb{K}$ into $R$. Then $\phi_R$ define a morphism $f_R$ of $(\operatorname{Spec} R, \mathcal{O}_R)$ to $(\operatorname{Spec} \mathbb{K}, \mathcal{O}_{\mathbb{K}})$. These $f_R$'s define a morphism $\pi_{\mathcal{X}} : (X, \mathcal{O}_X) \to (\operatorname{Spec} \mathbb{K}, \mathcal{O}_{\mathbb{K}})$, which is called the *structure morphism* of $\mathcal{X}$.

The *diagonal morphism* $\Delta_X$ is the unique morphism of $\mathcal{X}$ to $\mathcal{X}\times\mathcal{X}$ such that $p_1 \cdot \Delta_X = p_2 \cdot \Delta_X = 1_X$, where $p_1$ and $p_2$ are the projections of $\mathcal{X} \times \mathcal{X}$ to the first and the second factors of $X \times X$ respectively.

Let $\mathcal{G} = (G, \mathcal{O}_G)$ be a scheme over $\mathbb{K}$. Then we say that $(G, \mathcal{O}_G, \mu)$ is a *group scheme* over $\mathbb{K}$ if it satisfies the following conditions: (1) $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is such that $\mu(1_G \times \mu) = \mu(\mu \times 1_G)$; (2) there exists an endomorphism $\gamma$ of $G$ and a morphism $\epsilon$ of $\operatorname{Spec} R$ to $G$ such that $\mu(1_G \times \gamma)\Delta_G = \mu(\gamma \times 1_G)\Delta_G = \epsilon \cdot \pi_G$, where $\Delta_G$ and $\pi_G$ are the diagonal and the structure morphism of $G$ respectively; and (3) $\mu(\epsilon\times 1_G) = \mu(1_G\times\epsilon) = 1_G$, where we identify $\operatorname{Spec}(k) \times G$ and $G \times \operatorname{Spec}(k)$ with $G$ canonically. The morphisms $\mu$,

$\epsilon$ and $\gamma$ are called the multiplication, the identity morphism and the inverse morphism of $G$ respectively, and the image $e$ of $\epsilon$ in $G$ is called the neutral point of $G$. We denote a group scheme as $(G, \mu, \epsilon, \gamma)$. For example, $GL_n$ can be defined as the affine scheme of invertible integral $n \times n$ matrices, $\operatorname{Spec} \mathbb{Z}[x_{ij}][det(x_{ij})^{-1}]$.

Define $\operatorname{Mor}(X, G)$ as the set of all scheme-morphisms from the scheme $(X, \mathcal{O}_X)$ over $\mathbb{K}$ to the group scheme $(G, \mathcal{O}_G, \mu)$ over $\mathbb{K}$. Then $\operatorname{Mor}(X, G)$ forms a group under the composition $f * g = \mu(f \times g)\Delta_X$ for $f, g \in \operatorname{Mor}(X, G)$. Note that the base space $G$ of a group scheme $(G, \mathcal{O}_G, \mu)$ over $\mathbb{K}$ is not a group in general. Consider for example the polynomial ring $R = \mathbb{R}[x]$, $G = \operatorname{Spec} R$ which gives the additive group scheme $\mathbb{R}_a$ whereas its topological points or its closed points: $G = \{(0), (x - a), (x - z)(x - \bar{z}); a \in \mathbb{R}, z \in \mathbb{C}\}$ fail to form a group!

If $R$ is a commutative ring, then we have an equivalence of the category of $R$-group schemes with a full sub category of the category of functors from $R$-algebras to *Groups*. So we can also think of an $R$-group scheme $(G, \mathcal{O}_G)$ as a functor on $R$-algebras where $G(A) = \operatorname{Mor}(\operatorname{Spec}(A), G)$ is a group for any $R$-algebra, $A$.

**Examples.** For any commutative $\mathbb{K}$-algebra $A$, consider the following examples of a group scheme $\mathcal{G}$ as a functor on the category of $\mathbb{K}$-algebras,

(i) For a discrete group $G$, the constant functor $\mathcal{G}(A) = G$ defines a group scheme. The topological space is the topological group $G$ and $\mathcal{O}_p = \mathbb{K}$.

(ii) If $H$ is a finitely generated commutative Hopf Algebra, the group scheme $\mathcal{G}$ takes $A$ to the group of algebra morphisms from $H$ to $A$. Here the topological space is $\operatorname{Spec} H$ and $\mathcal{G}(A) = \operatorname{Hom}(H, A)$.

(iii) The group scheme $\mathcal{G} = GL_n$ associates to every $\mathbb{K}$-algebra $A$, the group $GL_n(A)$.

(iv) $\mathbb{C}^n/\mathbb{Z}^{2n}$ is a group scheme over $\mathbb{C}$.

### 2.5.1 Algebraic Groups

An affine *algebraic group* is an affine reduced group scheme $\mathbb{G}$ over algebraically closed field $\mathbb{K}$ equipped with morphisms of varieties $\mu : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$, $i : \mathbb{G} \to \mathbb{G}$ that give $\mathbb{G}$ the structure of a group. A morphism $f : \mathbb{G} \to \mathbb{H}$ of algebraic groups is a morphism of varieties that is a group homomorphism too.

**Examples.** (i) The additive group $\mathbb{G}_a$ i.e. the affine variety $\mathbb{A}^1$ under addition. (ii) The multiplicative group $\mathbb{G}_m$ i.e. the principal open subset $\mathbb{A}^1 \setminus \{0\}$ under multiplication. (iii) The group $\mathrm{GL}_n = \mathrm{GL}_n(k)$ of all invertible $n \times n$ matrices over $\mathbb{K}$. As a variety this is a principal open set in $M_n(\mathbb{K}) = \mathbb{A}^{n^2}$ corresponding to the determinant. (iv) The group $\mathrm{SL}_n = \mathrm{SL}_n(\mathbb{K})$ is the closed subgroup of $\mathrm{GL}_n$ defined by the zeros of $det - 1$.

Let $\mathbb{G}$ be an (affine) algebraic group with the identity element $e$. Let $H = \mathbb{K}[\mathbb{G}]$. Then the map

$$\epsilon : H \to k, f \mapsto f(e)$$

is an algebra homomorphism (called *augmentation*). Consider also the dual morphisms

$$\Delta := \mu^* : H \to H \otimes H$$

(called *comultiplication*) and

$$\sigma := i^* : H \to H$$

(called *antipode* ). It follows using the group axioms that these define the structure of a Hopf algebra on $\mathbb{K}[\mathbb{G}]$. Conversely, a structure of the Hopf algebra on $\mathbb{K}[\mathbb{G}]$ defines a structure of an algebraic group on $\mathbb{G}$. Infact, we have the following known fact:

**Theorem 2.5.2** *The categories of (affine) algebraic groups and affine Hopf algebras are contravariantly equivalent.*

Note that group schemes generalise algebraic groups, in the sense that all algebraic groups have a group scheme structure, but group schemes are not necessarily connected, smooth, or defined over a field. Infact, an algebraic group is a reduced group scheme over an algebraically closed field. For convenience we shall set following identification for the thesis: $\mathbb{G} = \mathbb{G}(\mathbb{K})$, where $\mathbb{G}$ on the right side is a reduced group scheme over an algebraically closed field $\mathbb{K}$, so that $\mathbb{G}(\mathbb{K})$ is a group. Moreover, $\mathbb{G}$ on the left side will be an algebraic group inheriting the variety structure from the group scheme $\mathbb{G}$.

## 2.6 Sheaves

Let $(X, \mathcal{O}_X)$ be an affine scheme with $X = \operatorname{Spec} R$, and a commutative ring $R$. Suppose $M$ is an $R$-module. We define $\mathcal{M}$ as a sheaf on the basic open set $X_x$ of $X$ as follows. To every $x \in X$ and the corresponding basic open set $X_x$, we associate $\mathcal{M}_x = M \otimes_R R_x$ ; the restriction map is the obvious one. Then $\mathcal{M}$ is a sheaf of $\mathcal{O}_{\operatorname{Spec} R}$ -modules. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on $X$. Then $\mathcal{F}$ is

(i) finitely generated(f.g.)/finite type if every point $x \in X$ has an open neighbourhood $U$ such that there is a surjective morphism

$$\mathcal{O}_X^n|_U \to \mathcal{F}|_U$$

where $n \in \mathbb{N}$.

(ii) finitely presented (f.p.) sheaf if there is an exact sequence of the form

$$\mathcal{O}_X^p|_U \to \mathcal{O}_X^n|_U \to \mathcal{F}|_U \to 0$$

where $p, n \in \mathbb{N}$.

Note. Every f.p. $\mathcal{O}_X$-module is f.g.

(iii) quasi-coherent sheaf if for every affine open $\operatorname{Spec} R$,

$$\mathcal{F}|_{\operatorname{Spec} R} \cong \Gamma(\widetilde{\operatorname{Spec} R}, \mathcal{F}).$$

(The wide tilde is supposed to cover the entire right side $\Gamma(\operatorname{Spec} R, \mathcal{F})$. This isomorphism $\operatorname{Spec} R$ is as sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is a quasi-coherent sheaf, then $\mathcal{F}$ is locally a cokernel of a morphism of free-modules. This means there is an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of $X$ such that for every $\alpha$ there exist $I_\alpha$ and $J_\alpha$ (not necessarily finite) and an exact sequence of sheaves of $\mathcal{O}_X$-modules of the form:

$$\mathcal{O}_X^{I_\alpha}|_{U_\alpha} \to \mathcal{O}_X^{J_\alpha}|_{U_\alpha} \to \mathcal{F}|_{U_\alpha} \to 0.$$

(iv) coherent if

- it is finitely generated,
- for any open set $U$ of $X$, every morphism $\mathcal{O}^p|_U \to \mathcal{F}|_U$ of $\mathcal{O}_X$-modules has a finitely generated kernel, where $p \in \mathbb{N}$.

Note that if $\mathcal{F}$ is locally finite then the condition of finite generation implies coherence and the other way also.

### 2.6.1 Tensor product

We first discuss the tensor product of sheaves which are defined on the same scheme. Suppose $\mathcal{X} = (X, \mathcal{O}_X)$ is a scheme and $\mathcal{F}$, $\mathcal{G}$ be sheaves on $\mathcal{X}$. Then the (internal)tensor product of $\mathcal{F}$ and $\mathcal{G}$ over $\mathcal{X}$ is a sheaf $\mathcal{F} \otimes_{\mathcal{X}} \mathcal{G}$ (or simply $\mathcal{F} \otimes \mathcal{G}$) on $\mathcal{X}$ such that the section of the sheaf at any affine open set $U = \operatorname{Spec}(R)$ is defined as $\mathcal{F}(U) \otimes_R \mathcal{G}(U)$. In general, if $\mathcal{F}$ and $\mathcal{G}$ are any two $\mathcal{O}_{\mathcal{X}}$-modules, we define the tensor product $\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{G}$ to be the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_{\mathcal{X}}(U)} \mathcal{G}(U)$. This is also an $\mathcal{O}_{\mathcal{X}}$-module.

We now describe *external tensor product* of sheaves which are defined on different schemes. Let $\mathcal{X} = (X, \mathcal{O}_X)$ and $\mathcal{Y} = (Y, \mathcal{O}_Y)$ be schemes. Suppose $\mathcal{F}$ and $\mathcal{G}$ are sheaves respectively on $\mathcal{X}$ and $\mathcal{Y}$. We want to construct a sheaf $\mathcal{F} \boxtimes \mathcal{G}$ on the product space $\mathcal{X} \times \mathcal{Y}$. Let $\pi_1 : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ and $\pi_2 : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}$ be the projection maps of schemes. Then external tensor product of $\mathcal{F}$ and $\mathcal{G}$ is the normal tensor product of the pullback sheaves $\pi_1^*\mathcal{F}$ and $\pi_2^*\mathcal{G}$. Note that this yields a sheaf on $\mathcal{X} \times \mathcal{Y}$. On stalks, there is a canonical bijection $(\mathcal{F} \boxtimes \mathcal{G})_{(x,y)} \to \mathcal{F}_x \boxtimes \mathcal{G}_y$. In particular, you see that if $\mathcal{F}$ and $\mathcal{G}$ are the sheaves of $\mathbb{K}$-valued continuous functions on $X$ resp. $Y$, then $\mathcal{F} \boxtimes \mathcal{G}$ is a rather small subsheaf of the continuous functions on $X \times Y$.

Suppose $\mathcal{G}$ is a group scheme and $\mathcal{A}$ is a sheaf on $\mathcal{G}$. Then there are three maps from $\mathcal{G} \times \mathcal{G}$ to $\mathcal{G}$ namely, the multiplication, the first projection $\pi_1$, and the second projection $\pi_2$,

$$\mu, \pi_1, \pi_2 : \mathcal{G} \times \mathcal{G} \to \mathcal{G}.$$

Let us denote the global sections of $\mathcal{A}$ as

$$\Gamma := \Gamma(\mathcal{G}, \mathcal{A}) = \mathcal{A}(\mathcal{G}).$$

Then we have the following result, which we use in the last section of Chapter 4.

**Lemma 2.6.1** *Suppose $\mathcal{A}$ is a sheaf over a scheme $\mathcal{G}$. If $\mathcal{A}$ is generated by global sections $\Gamma$, then $\mathcal{A} \boxtimes \mathcal{A}$ is generated by $\Gamma \times \Gamma$.*

Proof: Suppose $\mathcal{A}$ is generated by global sections. Then, the natural map

$$\Gamma \otimes \mathcal{O}_{\mathcal{G}} \longrightarrow \mathcal{A}$$

is a surjection. Now, since pullbacks are right exact, we have the following surjection

$$\pi_1^*(\Gamma \otimes \mathcal{O}_{\mathcal{G}}) = \Gamma \otimes \pi_1^* \mathcal{O}_{\mathcal{G}} = \Gamma \otimes \mathcal{O}_{\mathcal{G}\times\mathcal{G}} \longrightarrow \pi_1^* \mathcal{A}. \tag{2.15}$$

Taking tensor product is also right exact, so tensoring with $\pi_2^*\mathcal{A}$, (2.15) gives

$$f : \Gamma \otimes \pi_2^*\mathcal{A} = \Gamma \otimes \mathcal{O}_{\mathcal{G}\times\mathcal{G}} \otimes \pi_2^*\mathcal{A} \longrightarrow \pi_1^*\mathcal{A} \otimes \pi_2^*\mathcal{A} = \mathcal{A} \boxtimes \mathcal{A}.$$

Finally, taking tensor product with $\Gamma$, (2.15) gives the following surjection

$$g : \Gamma \otimes \Gamma \otimes \mathcal{O}_{\mathcal{G}\times\mathcal{G}} \longrightarrow \Gamma \otimes \pi_2^*\mathcal{A}.$$

Composing the last two maps, we get the required surjection

$$h = fg : \Gamma \otimes \Gamma \otimes \mathcal{O}_{\mathcal{G}\times\mathcal{G}} \to \mathcal{A} \boxtimes \mathcal{A}.$$

which makes $\mathcal{A} \boxtimes \mathcal{A}$ being generated by $\Gamma \otimes \Gamma$. □

### 2.6.2 $\mathbb{G}$-equivariant sheaves

Let $\mathcal{G}$ be a group scheme. Let $X$ be a $\mathcal{G}$-scheme i.e. a scheme equipped with an algebraic $\mathcal{G}$-action, $a : \mathcal{G} \times X \to X$. Thus there are two natural maps

$$\mathcal{G} \times X \underset{p}{\overset{a}{\rightrightarrows}} X$$

where $p$ is the projection map and $a$ is the action of $\mathcal{G}$ on $X$. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on an algebraic $\mathcal{G}$-variety $X$ is called $\mathcal{G}$-equivariant if the following conditions hold. [CG97].

**(a)** There is a *given* isomorphism of sheaves on $\mathcal{G} \times X$

$$I : a^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F}$$

**(b)** The pullbacks by $\mathrm{id} \times a$ and $m \times \mathrm{id}$ of the isomorphism $I$ are related by the equation $p_{23}^* I \circ (\mathrm{id}_{\mathcal{G}} \times a)^* I = (m \times \mathrm{id}_X)^* I$, where $p_{23} : \mathcal{G} \times \mathcal{G} \times X \to \mathcal{G} \times X$ is the projection along the first factor $\mathcal{G}$.

**(c)** For $e =$ the unit of $\mathcal{G}$, we have

$$I_{e\times X} = \mathrm{id} : \mathcal{F} = a^*\mathcal{F}|_{e\times X} \xrightarrow{\sim} p^*\mathcal{F}|_{e\times X} = \mathcal{F}$$

**Remark.** (i) For any $\mathcal{G}$-variety $X$, the sheaf $\mathcal{O}_X$ has canonical $\mathcal{G}$-equivariant structure given by the composition of the following natural isomorphisms: $p^*\mathcal{O}_X \simeq \mathcal{O}_{\mathcal{G}\times X} \simeq a^*\mathcal{O}_X$.

(ii) Condition (c) in the definition above is superfluous and is only given for convenience. Indeed, it can be deduced from (a) and (b) as follows. Restricting the equality in (b) to $e \times e \times X$, one finds $I_{e\times X} \circ I_{e\times X} = I_{e\times X}$. Since $I_{e\times X}$ is an isomorphism, this yields $I_{e\times X} = \mathrm{id}$.

### 2.6.3 Cosheaves

Let $R$ denote a ring. Let $X$ be a topological space. A precosheaf $\mathcal{C}$ (of $R$-modules) on $X$ is a covariant functor from the category of open sets in $X$ and inclusions to the category of $R$-modules, [Bre97]. For $U \subseteq V$ the corresponding map $\mathcal{C}(U) \to \mathcal{C}(V)$ called *corestrictions* is denoted by $i_{V,U}$.

A precosheaf $\mathcal{C}$ on $X$ is called a cosheaf if, for every open covering $\{U_\alpha\}$ of an open set $U \subseteq X$, the sequence

$$\bigoplus_{<\alpha,\beta>} \mathcal{C}(U_\alpha \cap U_\beta) \xrightarrow{\varphi} \bigoplus_{\alpha} \mathcal{C}(U_\alpha) \xrightarrow{\phi} \mathcal{C}(U) \to 0$$

is exact, where $\varphi = \sum_\alpha i_{U,U_\alpha}$ and $\phi = (\sum_{<\alpha,\beta>} i_{U_\alpha,U_\alpha\cap U_\beta} - i_{U_\beta,U_\alpha\cap U_\beta})$. The following result states the necessary and sufficient condition for a precosheaf to become a cosheaf.

**Proposition 2.6.2** *[Bre97] Let $\mathcal{C}$ be a precosheaf. Then $\mathcal{C}$ is a cosheaf if and only if the* *following two conditions are satisfied:*

**(a)** $\mathcal{C}(U \cap V) \xrightarrow{\varphi} \mathcal{C}(U) \oplus \mathcal{C}(V) \xrightarrow{\phi} \mathcal{C}(U \cup V) \to 0$ *is exact for all open* $U$ *and* $V$ *where* $\varphi = (i_{U,U\cap V} - i_{V,U\cap V})$ *and* $\phi = (i_{U\cap V,U} + i_{U\cap V,V})$.

**(b)** *If* $\{U_\alpha\}$ *is directed upwards by inclusion then the natural map lim* $\mathcal{C}(U) \to \mathcal{C}(\cup_\alpha U_\alpha)$ *is an isomorphism.*

Let us discuss the local nature of the notion of a cosheaf formulated for any precosheaf on a topological space $X$. A precosheaf $\mathcal{C}$ on a topological space $X$ is a cosheaf if for any open set $U \subseteq X$ and any any open covering $U = \cup U_\alpha$ of $U$, the following two conditions hold:

(i) For any $u \in \mathcal{C}(U)$, there exists finitely many $u_{\alpha_i} \in \mathcal{C}(U_{\alpha_i})$ such that $\sum i_{U,U_{\alpha_i}}(u_{\alpha_i}) = u$ for each $U_{\alpha_i}$.

(ii) For all $u_\alpha \in \mathcal{C}(U_\alpha)$, such that $\varphi(u_\alpha) = 0$ there exist $u_{\alpha\beta} \in \mathcal{C}(U_\alpha \cap U_\beta)$ such that $\phi(u_{\alpha\beta}) = u_\alpha$.

**Examples.** Let $X = \{x_1, x_2\}$ with the discrete topology, i.e. all subsets are open. We write $X_i = \{x_i\}$. We define precosheaves on $X$ as follows.

(i)

$$\begin{aligned} \mathcal{C}_1 : \mathcal{C}_1(X_i) &= \mathbb{Z} \quad (1 \le i \le 2) \\ \mathcal{C}_1(X) &= \mathbb{Z} \\ i^{\mathcal{C}_1}_{U,V} &= 0 \quad \text{for } U \neq V. \end{aligned}$$

Clearly $\mathcal{C}_1$ is a precosheaf. But $\mathcal{C}_1$ fails to be a cosheaf because condition (i) in the above definition fails: $2 \in \mathcal{C}_1(X)$, but there is no $u_1$ in $X_1$ such that $i_{X,X_1}(u_1) = 2$ since $2 \neq 0$.

(ii)

$$\begin{aligned}
\mathcal{C}_2 : \mathcal{C}_2(X_i) &= \mathbb{Z} \quad (1 \leq i \leq 2) \\
\mathcal{C}_2(X) &= \mathbb{Z} \\
i^{\mathcal{C}_2}_{U,V} &= \mathrm{id}_{\mathbb{Z}} \quad \text{for } V \neq \emptyset.
\end{aligned}$$

Clearly $\mathcal{C}_2$ is a precosheaf. Moreover, $\mathcal{C}_2$ is also a cosheaf.

(iii)

$$\begin{aligned}
\mathcal{C}_3 : \mathcal{C}_3(X_i) &= \mathbb{Z} \oplus \mathbb{Z} \quad (1 \leq i \leq 2) \\
\mathcal{C}_3(X) &= \mathbb{Z} \\
i^{\mathcal{C}_3}_{X,X_1} &= \pi_1 : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \;\; : (m,n) \mapsto m \\
i^{\mathcal{C}_3}_{X,X_2} &= \pi_1 : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \;\; : (m,n) \mapsto n.
\end{aligned}$$

Then $\mathcal{C}_3$ is a precosheaf. But $\mathcal{C}_3$ fails to be a cosheaf because condition (ii) in the above definition fails: For $u_1 = (2,3) \in \mathcal{C}_3(X_1)$, and $u_2 = (1,-2) \in \mathcal{C}_3(X_2)$, $\phi(\sum_{i=1}^{2} u_i) = (i_{U,U_1} + i_{U,U_2})(u_1 + u_2) = 2 - 2 = 0$ but there are no $u_{12}, u_{12}^{'} \in \mathcal{C}_3(X_1 \cap X_2) = 0$ such that $\left(i_{X_1,\emptyset} - i_{X_2,\emptyset}\right)(u_{12} + u_{12}^{'}) = (2,3) + (1,-2) = (3,1)$ since $i_{X_1,\emptyset}(x)$ is a zero map and $(3,1) \neq (0,0)$.

Let $\mathcal{A}$ be a sheaf over a scheme $\mathcal{G}$. Let us discuss its dual cosheaf denoted by $\mathcal{A}^*$. It is defined as

$$\mathcal{A}^* = \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, \mathbb{K}).$$

Then for each open covering $\{U_i\}$ of $U \subseteq \mathcal{G}$, the sheaf exact sequence of $\mathbb{K}$-modules,

$$\prod_{i,j} \mathcal{A}(U_i \cap U_j) \leftleftarrows \prod_{i} \mathcal{A}(U_i) \longleftarrow \mathcal{A}(U) \longleftarrow 0$$

turns into cosheaf exact sequence of $\mathbb{K}$-modules

$$\oplus_{i,j} \mathcal{A}^*(U_i \cap U_j) \rightrightarrows \oplus_i \mathcal{A}^*(U_i) \longrightarrow \mathcal{A}^*(U) \longrightarrow 0.$$

Thus $\mathcal{A}^*$ is a cosheaf of $\mathbb{K}$-modules over group scheme $\mathcal{G}$. Let us now discuss the finite dual of $\mathcal{A}$ which is given by

$$\mathcal{A}^\circ = \Big\{ f \in \mathcal{A}^* \Big| f(I) = 0 \text{ for some ideal } I \text{ of } \mathcal{A} \text{ ; } \dim_{\mathbb{K}} \mathcal{A}/I < \infty \Big\}. \tag{2.16}$$

Then for any $f \in \mathcal{A}^\circ(U)$ if ${}_fM = (\mathcal{A}(U) \rightharpoonup f)$ and $M_f = (\mathcal{A}(U) \leftharpoonup f)$, we have that $\dim_{\mathbb{K}}({}_fM)$, $\dim_{\mathbb{K}}(M_f) < \infty$. Moreover, $F$ factors through ${}_fM$ and $M_f$, that is,

$$f : \mathcal{A}(U) \to {}_fM \to \mathbb{K},$$

$$f : \mathcal{A}(U) \to M_f \to \mathbb{K}.$$

Cosections of $\mathcal{A}^\circ$ are subspaces of the corresponding cosections of $\mathcal{A}^*$. For each inclusion of open sets $U \subseteq V$, the corestriction $cores^\circ_{V,U} : \mathcal{A}^\circ(U) \to \mathcal{A}^\circ(V)$ is given by the following sequence

$$cores^\circ_{V,U}(f) : \mathcal{A}(V) \xrightarrow{res_{U,V}} \mathcal{A}(U) \longrightarrow {}_fM \longrightarrow \mathbb{K},$$

Thus the corestrictions of the cosheaf $\mathcal{A}^\circ$ are induced by the restrictions of $\mathcal{A}^*$. This makes $\mathcal{A}^\circ$ a subcosheaf of $\mathcal{A}^*$.

Observe that we can think of sheaves and cosheaves in terms of categories in the following sense:

(i) A sheaf $\mathcal{F}$ on a topological space $X$ can be thought of as a contravariant functor from the category of open sets on $X$ to some abelian category $A$ with infinite products, such that they satisfy exactness condition in $A$.

$$\mathcal{F} : \mathrm{Open}_X \xrightarrow{\mathcal{F}} A.$$

(ii) A cosheaf $\mathcal{C}$ on a topological space $X$ can be thought of as a covariant functor from the category of open sets on $X$ to an abelian category $A$ with infinite sums, such that they satisfy exactness condition in $A$.

$$\mathcal{C} : \mathrm{Open}_X \xrightarrow{\mathcal{C}} A.$$

(iii) The dual of a sheaf $\mathcal{F}$ becomes a covariant functor $\mathcal{G}$ :

$$\mathcal{G} : \mathrm{Open}_X \xrightarrow{\mathcal{F}} A \xrightarrow{\mathcal{D}} A'$$

where $\mathcal{D} : A \to A'$ is a contravariant exact functor such that $\mathcal{D}$ takes sheaves in $A$ into cosheaves in $A$, and vice-versa. Since composition of two contravariant functor is a covariant functor, the dual of a sheaf is a cosheaf.

(iv) If $\mathcal{F}$ is a sheaf of vector spaces on a topological space $X$ then the dual of $\mathcal{F}$ is a cosheaf of vector spaces on the same topological space $X$. Note that dual of a vector space is a vector space and the dual of a linear map is linear in addition to duality being exact.

(v) If $\mathcal{F}$ is a sheaf on $X$ of finite dimensional $\mathbb{K}$-algebras then the dual sheaf $\mathcal{F}^*$ is a cosheaf on $X$ of $\mathbb{K}$-coalgebras.

Analogous to the theory of sheaves over a topological space, we have some more definitions to say for cosheaves. If $\mathcal{C}$ is a cosheaf on $X$, and $x \in X$, then we define *costalk* $\mathcal{C}_x$ of $\mathcal{C}$ at $x$ to be the inverse(projective) limit of the sets $\mathcal{C}(U)$ taken over all open sets $U \ni x$ with respect to the system of map $i_{V,U}$ for $U \subset V$. In the general case, for any open set $U \ni x$, there exists a natural homomorphism of groups

$$i_{U,x} : \mathcal{C}_x \longrightarrow \mathcal{C}(U).$$

Let $x \in U \subseteq X$ and $u_x \in \mathcal{C}_x$ such that $i_{U,x}(u_x) \in \mathcal{C}(U)$. Then for any neighbourhood $W$ of $x$ such that $x \in W \subseteq U$, there exists $w \in \mathcal{C}(W)$ such that

$$i_{U,x}(u_x) = i_{U,W}(w).$$

Note that here $w = i_{W,x}(u_x)$. Further, if $u \in \mathcal{C}(U)$, then there exists finitely many $u_{x_i} \in \mathcal{C}_{x_i}$ such that $\sum i_{U,U_{x_i}}(u_{x_i}) = u \ \forall x_i \in U_{x_i}$. Thus for a cosheaf $\mathcal{C}$, the elements of $\mathcal{C}(U)$ can be specified as families $\{u_{x_i}\}_{x_i \in U_{x_i}}$. We shall call elements of $\mathcal{C}(U)$ the cosections.

**Example.** As we have given $X$ the discrete topology,

$$\mathcal{C}_{x_i} = \mathcal{C}(X_i) \quad (1 \leq i \leq 2)$$

If $\mathcal{C}$ and $\mathcal{D}$ are cosheaves on $X$, a *morphism of cosheaves* $\phi : \mathcal{C} \to \mathcal{D}$ consists of a morphism of abelian groups $\phi(U) : \mathcal{C}(U) \to \mathcal{D}(U)$ for each open set $U$, such that whenever $V \subseteq U$ is an inclusion, the diagram

$$\begin{tikzcd}
\mathcal{C}(V) \arrow[r, "\phi(V)"] \arrow[d, "i^{\mathcal{C}}_{U,V}"'] & \mathcal{D}(V) \arrow[d, "i^{\mathcal{D}}_{U,V}"] \\
\mathcal{C}(U) \arrow[r, "\phi(U)"] & \mathcal{D}(U)
\end{tikzcd}$$

is commutative, where $i^{\mathcal{C}}$ and $i^{\mathcal{D}}$ are the corestriction maps in $\mathcal{C}$ and $\mathcal{D}$ respectively. An isomorphism is morphism which has two sided inverse. Note that a morphism $\phi : \mathcal{C} \to \mathcal{D}$ of precosheaves on $X$ induces a morphism $\phi_x : \mathcal{C}_x \to \mathcal{D}_x$ on the costalks, for any point $x \in X$.

A subcosheaf of a cosheaf $\mathcal{F}$ over $X$ is a cosheaf $\mathcal{F}^{'}$ such that for every open set $U \subseteq X$, $\mathcal{F}^{'}(U)$ is a subgroup of $\mathcal{F}(U)$, and the restriction maps of the sheaf $\mathcal{F}^{'}$ are induced by those of $\mathcal{F}$. It follows that for any point $x$ in $X$, the stalk $\mathcal{F}^{'}_x$ is a subgroup of $\mathcal{F}^{'}$.

Let $f : X \to Y$ be a continuous map of toplogical spaces. For any cosheaf $\mathcal{F}$ on $X$, we define *direct image* cosheaf $f_*\mathcal{F}$ on $Y$ by $f_*\mathcal{F}(V) = \mathcal{F}\big(f^{-1}(V)\big)$ for any open set $V \subseteq Y$.

## 2.7 Disjoint Union

Let $M, N$ be any two sets. We define their disjoint union as a set containing all the elements of $M$ and $N$ but now they are all labelled. Explicitly,

$$M \sqcup N = \big(M \times \{0\}\big) \cup \big(N \times \{1\}\big) \subseteq \big(M \cup N\big) \times \{0, 1\}.$$

**Remarks.**

(i) Note that $M$ and $N$ are not subsets of their disjoint union. but we certainly have injective maps $M \hookrightarrow M \sqcup N$ and $N \hookrightarrow M \sqcup N$. These maps simply put the relevant index on each element, for example $x \longmapsto (x, 1)$

(ii) The way disjoint union is defined, it satisfies the universal property making it the coproduct in the category of sets.

(iii) Clearly, $M \sqcup M \neq M$ and $M \sqcup \emptyset \neq M$ or $\emptyset \sqcup M \neq M$. Moreover, for any $M \neq N$, $M \sqcup N \neq N \sqcup M$ as for any $m \in M$ such that $m \notin N$, $(m, 0) \in M \sqcup N$ but $(m, 0) \notin N \sqcup M$. (If $N = M$, then of course they are the same).

Similarly we can define the disjoint union of $n$-factors. Note again that $(L \sqcup M) \sqcup N \neq L \sqcup (M \sqcup N)$ and so on. With the disjoint union defined on sets, $(\mathrm{Set}, \sqcup, \emptyset)$ has a symmetric monoidal structure. The associativity is just a matter of relabelling, and thus there is a bijection of sets:

$$a : (L \sqcup M) \sqcup N \xrightarrow{\sim} L \sqcup (M \sqcup N)$$

such that $(l, 0, 0) \sqcup (m, 0, 1) \sqcup (n, 1) \mapsto (l, 0) \sqcup (m, 1, 0) \sqcup (n, 1, 1)$ for $l \in L$, $m \in M$ and $n \in N$. In fact, we get a whole family of maps : one for each triplet of sets. This family is natural in the category of sets. The left and right units are defined by the obvious bijections $(x, 1) \mapsto x$ and $(x, 0) \mapsto x$ respectively. Again, interchanging the labelling gives a bijection of sets

$$c_{M,N} : M \sqcup N \xrightarrow{\sim} N \sqcup M$$

called the twist map. We again get a family of maps : one for each pair of sets. This family is also natural in the category of sets. An important property of the twist map is

$$c_{M,N} \circ c_{N,M} = id_{M \sqcup N}.$$

For sets $K, L, M, N$, consider any pair of morphisms $f : K \to M$ and $g : L \to N$. Then their disjoint union(coproduct) $f \sqcup g : K \sqcup L \to M \sqcup N$ is defined as

$$(x, 0) \mapsto \big(f(x), 0\big) \text{ and } (x, 1) \mapsto \big(g(x), 1\big)$$

for any $x \in K \cup L$. Thus $(\text{Set}, \sqcup, \emptyset)$ is a symmetric monoidal category.

## 2.8 Orientation

For this section, manifold means a compact, topological manifold possibly with boundary.

**Theorem 2.8.1** *[Spa81] Let $M$ be a connected, $n$-dimensional manifold. Then $H_n(M, \partial M; \mathbb{Z})$, the top dimensional homology group of $M$, is either trivial or isomorphic to $\mathbb{Z}$.*

**Definition 2.8.2** *A connected, $n$-manifold $M$ is called orientable if its top homology group is isomorphic to the integers. An orientation of $M$ is a choice of a particular isomorphism*

$$\mathfrak{o} : \mathbb{Z} \to H_n(M, \partial M; \mathbb{Z})$$

A *fundamental class* of $M$, denoted by $[M]$ is a generator of the $\mathbb{Z}$-module $H_n(M, \partial M, \mathbb{Z})$. Thus a fundamental class of $M$ gives an orientation of $M$ by $\mathfrak{o}(1) = [M]$.

Now suppose $M$ is a disconnected $n$-dimensional manifold with $\{M_{(i)}\}$ the connected components of $M$. An orientation of $M$ is given by a choice of orientations of its connected components:

$$H_n(M, \partial M; \mathbb{Z}) \cong \bigoplus_i H_n(M_{(i)}, \partial M_{(i)}; \mathbb{Z})$$

so that restriction to each component is an orientation of that component. The fundamental class $[M]$ is given by $([M_{(i)}])$, where $[M_{(i)}]$ is the fundamental class of its connected component $M_{(i)}$.

Suppose $M$ is an $n$-manifold with a non-empty boundary $\partial M$. Then $\partial M$ is a compact $(n-1)$-manifold without boundary. Suppose now $M$ is orientable with a non-empty boundary, then its boundary is also orientable. Let us explain the induced orientation on the boundary. The theorem below discusses the induced orientation on the boundary of an $n$-manifold. The proof has been discussed in [Spa81]:

**Theorem 2.8.3** *[Spa81] If $M$ is a compact $n$-manifold with boundary $\partial M$, then if $M$ is orientable, so is $\partial M$, and any fundamental class of $M$ maps to a fundamental class of*

*$\partial M$ under the connecting homomorphism*

$$\partial_* : H_n(M, \partial M; \mathbb{Z}) \to H_{n-1}(\partial M; \mathbb{Z}).$$

Before we discuss orientation for gluing, let us briefly explain gluing of $n$-manifolds. Suppose $W_1$ and $W_2$ are any two $n$-dimensional manifolds with non-empty boundaries:

$$\partial W_1 = K_1 \dot{\cup} L$$

$$\partial W_2 = L^{'} \dot{\cup} K_2.$$

Let $\varphi : L \to L^{'}$ be a homeomorphism of manifolds. Then we can glue $W_1$ and $W_2$ along $\varphi$. Let $f_1 : L \to W_1$ and $f_2 : L^{'} \to W_2$ be continuous maps. Then,

$$W = W_1 \sqcup W_2 / \varphi$$

is defined by taking the disjoint union of $W_1$ and $W_2$ and quotienting out by the equivalence relation given by $\varphi$: two points $w_1 \in W_1$ and $w_2 \in W_2$ are equivalent in $W$ if there is a point $x \in L$ such that $f_1(x) = w_1$ and $f_2\big(\varphi(x)\big) = w_2$. Note that $W_1 \sqcup W_2$ has disjoint union topology and $W$ gets a quotient topology. Observe that $L$ and $L^{'}$ are no longer a boundary of $W$. Indeed, $W$ is an $n$-manifold with boundary $K_1 \dot{\cup} K_2$. There are two natural maps $W_1 \longrightarrow W \longleftarrow W_2$. Now suppose $W_1$ and $W_2$ are oriented $n$-manifolds. We wish to provide an orientation to $W$. We formulate the following theorem to express the orientation on $W$.

**Theorem 2.8.4** *Suppose $W_1$, $W_2$ are oriented $n$-manifolds with boundaries*

$$\partial W_1 = K \dot{\cup} L$$

$$\partial W_2 = L \dot{\cup} M$$

*such that $K \cap M = \emptyset$ and $W = W_1 \cup W_2$. If $L$ induces opposite orientation when considered as a boundary of $W_1$ and $W_2$ respectively, then the orientation of $W_1$ and $W_2$*

*can be extended to provide an orientation on $W$.*

PROOF: We do the proof in three easy steps. Firstly, observe that the relative Mayor Vietoris sequence

$$\cdots \to H_n(L,L) \to H_n(W_1,\partial W_1)\oplus H_n(W_2,\partial W_2) \to H_n(W,K\cup L\cup M) \to H_{n-1}(L,L) \to \cdots$$

for an excisive couple of pairs $\{(W_1,\partial W_1),(W_2,\partial W_2)\}$ yields an isomorphism

$$H_n(W_1,\partial W_1)\oplus H_n(W_2,\partial W_2) \cong H_n(W,K\cup L\cup M)$$

since $H_n(L,L) = H_{n-1}(L,L) = 0$. Secondly, the excision map

$$(L,\emptyset) \subseteq (K\cup L\cup M, K\cup M)$$

induces an isomorphism on homology

$$H_n(L) \cong H_n(K\cup L\cup M, K\cup M).$$

Finally, the inclusion of the pairs

$$(K\cup L\cup M, K\cup M) \subseteq (W, K\cup M) \subseteq (W, K\cup L\cup M)$$

yield a sequence of homomorphism (all coefficients in $\mathbb{Z}$)

$$H_n(L) \to H_n(W,\partial W) \xrightarrow{\alpha} H_n(W_1,\partial W_1)\oplus H_n(W_2,\partial W_2) \xrightarrow{\partial_*} H_{n-1}(L).$$

Note that $H_n(L) = 0$ and if $[W_1]$ and $[W_2]$ are fundamental classes of $W_1$ and $W_2$ respectively, then the given condition implies that $\partial_*\big([W_1]+[W_2]\big) = 0$. Then $[W_1]+[W_2] = \alpha(x)$ for some $x \in H_n(W,\partial W)$. It is easy to show that $x$ is a fundamental class of $W$. Hence the orientation of $W_1$ and $W_2$ is extended to provide an orientation on $W$. □

## 2.9 HQFT

Homotopy quantum field theories (HQFTs) were introduced by Turaev, and they are essentially TQFTs in a background space $X$, up to homotopy. A modified version with change to one of the axioms was introduced by Rodrigues, [Rod01]. This gave dependence of $(d+1)$-HQFTs over $X$ on the $(d+1)$-type of $X$. This was used by Brightwell and Turner, [BT00], and Turner and Willerton, [BTW02], to look at (1+1)-HQFTs with background space a simply connected space. Thus the results of [Tur99] had classified (1+1)-HQFTs with background spaces which were 1-types and the more recent results handled simply connected spaces, classification results there being in terms of the second homotopy group of $X$. Rodrigues showed that an $n$-dimensional HQFT can be regarded as a monoidal functor

$$Z : nCob(X) \to Vect$$

where $nCob(X)$ is the category whose objects are $(n-1)$ closed manifolds equipped with a map into $X$, and whose morphisms are cobordisms equipped with a map into $X$, considered up to homotopy (in $X$) fixing the boundary. We have adopted the same philosophy, but we work in topological setup whereas he works with diffeomorphisms of manifolds. In his setup, he considers diffeomorphisms between manifolds. The morphisms between objects in his category $nCob(X)$ are essentially strings of cobordisms and diffeomorphisms. On the other hand, our mechanism shall regard an $n$-dimensional HQFT as a monoidal functor

$$Z : \mathcal{X} - \mathrm{Cob}_n \to \mathcal{A}$$

for any monoidal category $\mathcal{A}$. The construction of our category $\mathcal{X} - \mathrm{Cob}_n$ involves going up to some level of 2-categories for defining a morphism between objects in $\mathcal{X} - \mathrm{Cob}_n$. The second difference is that his definition of an HQFT is stronger than the one given by Turaev. He has added structural isomorphisms which he demands to be natural *also* for cobordisms. We, on the other hand, work with the same definition of an HQFT as given by Turaev.

So one can think of HQFT as a midway between an abstract TQFT (with no background space) and a Stolz-Teichner style smooth TQFT embedded in $X$. Some authors have studied and discussed lower dimensional HQFTs in their own set up and have tried to compare them with something already known and hence tried to classify them. For instance if $X$ is a $K(\pi, 1)$ for some group $\pi$, then Turaev has shown that a $2d$ HQFT in $X$ is the same thing as a crossed $\pi$-algebra. A crossed $\pi$-algebra can be thought of as a Frobenius algebra object in $\mathrm{Rep}\,\Lambda\pi$- the category of representations of the loop groupoid of $\pi$. Similarly Brightwell and Turner showed that if $X$ is $K(A, 2)$, then a 2d-HQFT over $X$ is the same thing as a Frobenius algebra equipped with an action of $A$ which is a Frobenius algebra object in $\mathrm{Rep}\,A$. Then T.Porter along with Turaev [PT08] have extended these results to all 2-types. We know that a 2-type corresponds algebraically to a crossed module, so one begins by fixing a crossed module and working from there.

From a geometrical point of view a $1+1$-dimensional homotopy quantum field theory has been related to a vector bundle on the free loop space of $X$ with a generalised at connection, giving parallel transport across surfaces. Gerbes with connection have been characterised as functors on a certain surface cobordism category, [BTW04]. This has allowed the authors to relate gerbes with connection to Turaevs 1+1-dimensional homotopy quantum field theories, and they have shown that flat gerbes are related to a specific class of rank one homotopy quantum field theories.

To give the definition of an HQFT we first need to set up some background which has also been discussed by Turaev, [Tur99]

### 2.9.1 Preliminaries on HQFTs.

A locally connected topological space is *pointed* if all its connected components are provided with base points. A map between pointed spaces is a continuous map sending base points to base points. We shall work in the topological setup. Thus, by manifolds we shall mean topological manifolds. Let $X$ be a path connected topological space with base point $x \in X$. Later we will require this space to be Eilenberg MacLane space of type $K(\pi, 1)$ for some discrete group $\pi$. An $X$-$manifold$ is a pair $(M, g_M)$, where $M$ is a manifold such that every component of $M$ is a pointed closed oriented manifold and $g_M : M \to X$ is a map into $X$. This map is called the *characteristic map.* It sends the base points of all the components of $M$ into $x$. Note that an empty set $\emptyset$ is an $X$-manifold of any given dimension as there are no components of $\emptyset$ which are not pointed closed oriented. It is easy to see that the disjoint union of $X$-manifolds is an $X$-manifold. An *$X$-homeomorphism* of $X$-manifolds $f : M \to M^{'}$ is an orientation preserving homeomorphism sending base points of $M$ onto those of $M^{'}$ and such that $g_M = g_{M'} f$ where $g_M, g_{M'}$ are the characteristic maps of $M, M^{'}$ respectively.

An $n+1$ dimensional *cobordism* is a triple $(W, M_0, M_1)$ where $W$ is a compact oriented manifold whose boundary is a disjoint union of pointed closed oriented $n$-dimensional manifolds $M_0, M_1$ such that the orientation of $M_1$ (respectively $M_0$) is induced by the one of $W$ (respectively is opposite to the one induced from $W$). Note that the manifold $W$ itself is not required to be pointed.

An *$X$-cobordism* is a cobordism $(W, M_0, M_1)$ equipped with a map $W \to X$ sending the base points of the boundary components into $x$. Here both $M_0$ and $M_1$ are considered as $X$-manifolds with characteristic maps obtained by restricting the given map $W \to X$. An *$X$-homeomorphism of $X$-cobordisms* $f : (W, M_0, M_1) \to (W, M_0^{'}, M_1^{'})$ is an orientation preserving homeomorphism including $X$-homeomorphisms $M_0 \to M_0^{'}$, $M_1 \to M_1^{'}$ and such that $g_W = g_{W'} f$ where $g_W, g_{W'}$ are the characteristic maps of $W, W^{'}$, respectively.

One glue two $X$-cobordisms along the bases. If $(W_0, M_0, N), (W_1, N^{'}, M_1)$ are $X$-cobordisms and $f : N \to N^{'}$ is an $X$-homeomorphism then the gluing of $W_0$ to $W_1$ along $f$ yields a new $X$-cobordism with bases $M_0$ and $M_1$. Here it is essential that $g_N = g_{N'} f$.

### 2.9.2 Definition of an HQFT

Turaev gives the axiomatic definition of an HQFT with target $X$ using a version of Atiyah's axioms for a TQFT. Here an HQFT will take values in a monoidal category, $\mathcal{C}$. We shall be using disjoint union($\sqcup$) and the reader is referred to the Section (2.7) where we have discussed it in detail.

An $(n+1)$-dimensional $X$-HQFT $(\mathcal{Z}, \tau)$ assigns to any $n$-dimensional $X$-manifold $(M, g_M)$ an object $Z_M$ in $\mathcal{C}$, an isomorphism $f_\# : Z_M \to Z_N$ to any $X$-homeomorphism of $n$-dimensional $X$-manifolds $f : M \to N$ and to any $(n+1)$-dimensional $X$-cobordism $(W, M_0, M_1)$, a morphism $\tau(W) : Z_{M_0} \to Z_{M_1}$ in $\mathcal{C}$. These objects and morphisms satisfy the following eight axioms.

**Axiom (i)** If $f : M \to M^{'}$ and $f : M^{'} \to M^{''}$ are any two composable $X$-homeomorphism of $n$-dimensional $X$-manifolds, then $(f^{'}f)_{\#} = f^{'}_{\#}f_{\#}$.

**Axiom (ii)** For any disjoint $n$-dimensional $X$-manifolds $M, N$, there is a natural isomorphism $Z_{M\sqcup N} \cong Z_M \otimes Z_N$, where $\otimes$ is the monoidal structure in $\mathcal{C}$.

**Axiom (iii)** $Z_{\emptyset} \cong I_{\mathcal{C}}$.

**Axiom (iv)** Given any $X$-homeomorphism of $X$-cobordisms

$$F : (W, M_0, M_1, g) \to (W^{'}, M_0^{'}, M_1^{'}, g^{'}),$$

the following diagram

$$\begin{tikzcd}
Z_{(M_0, g|_{M_0})} \arrow[r, "(F|_{M_0})_{\#}"] \arrow[d, "\tau(W,g)"'] & Z_{(M_0^{'}, g^{'}|_{M_0^{'}})} \arrow[d, "\tau(W^{'},g^{'})"] \\
Z_{(M_1, g|_{M_1})} \arrow[r, "(F|_{M_0})_{\#}"'] & Z_{(M_1^{'}, g^{'}|_{M_1^{'}})}
\end{tikzcd}$$

commutes.

**Axiom (v)** If an $(n+1)$ dimensional $X$-cobordism $W$ is a disjoint union of two $X$-cobordisms representatives $W_1, W_2$, then $\tau(W) = \tau(W_1) \otimes \tau(W_2)$.

**Axiom (vi)** Gluing axiom. If an $X$-cobordism $(W, M_0, M_1)$ is obtained by gluing of $(W, M_0, N)$ and $(W, N^{'}, M_1)$ along an $X$-homeomorphism $f : N \to N^{'}$ then

$$\tau(W) = \tau(W_1) \circ f_{\#} \circ \tau(W_0) : Z_{M_0} \to Z_{M_1}.$$

**Axiom (vii)** For any $n$-dimensional $X$-manifold $(M, g_M)$, we have

$$\tau\Big(M \times [0,1], M \times 0, M \times 1, \bar{g}\Big) = id : Z_M \to Z_M,$$

where $M \times 0$ and $M \times 1$ are identified with $M$ in the usual way and where $\bar{g}$ is the composition of the projection $M \times [0,1] \to M$ with $M \to X$.

**Axiom (viii)** For any $(n+1)$-dimensional $X$-cobordism $W = (W, g : W \to X)$, the homomorphism $\mathcal{Z}(W)$ is preserved under any homotopy of $g$ relative to $\partial W$.

This definition has been given by Turaev [Tur99] and he calls it an HQFT with target $X$. But the one we discuss here have slight variations from his original definition given in [Tur99]. Firstly, his HQFT takes values in the category of projective modules of finite type over a commutative ring $K$ and $K$-linear homomorphisms. We have replaced this category by any general monoidal category $\mathcal{C}$. Thus $Z_M$ is an object of $\mathcal{C}$ for a $n$-dimensional $X$-manifold $(M, g_M)$ and $\tau(W)$ is a morphism in $\mathcal{C}$ for a $(n+1)$-dimensional $X$-cobordism $(W, g)$. Secondly, axiom (i) is a weekend version of the corresponding axiom in [Tur99]. He also requires that the isomorphism $f_\# : A_M \to A_{M'}$ is invariant under isotopies of $f$ in the class of $X$-homeomorphisms. Finally, his axiom (vii) asks:

For any $n$-dimensional $X$-manifold $(M, g : M \to X)$ and any map $F : M \times [0,1] \to X$ such that $F|_{M\times 0} = F|_{M\times 1} = g$ and $F(m \times [0,1]) = x$ for all base points $m \in M$, we have $\mathcal{Z}(M \times [0,1], F) = id : A_M \to A_M$ where the cylinder $M \times [0,1]$ is viewed as an $X$-cobordism with bases $M \times 0 = M$, $M \times 1 = M$ and characteristic map $F$."

**Remarks.**

(i) The Axioms (i)-(vii) constitute the standard definition of a TQFT.

(ii) In case a $(n+1)$-dimensional HQFT has target space as $X$={point}, then it is simply a TQFT of same dimension.

(iii) By Axiom (viii), $\tau(W)$ is a homotopy invariant of $g$. Any closed oriented $(n+1)$-dimensional $X$-manifold $W$ endowed with a map $g : W \to X$ can be considered as an $X$-cobordism with empty bases. Thus we get the corresponding endomorphism of $A_\emptyset = I_{\mathcal{C}}$

(iv) In case $\mathcal{A}$ is a category of vector spaces over a field $\mathbb{K}$, then for every $n$-dimensional manifold $M$, a $(n+1)$-dimensional HQFT gives a representation of the mapping class group of $M$.

(v) The naturality condition in Axiom (ii) means that the isomorphism $Z_{M \sqcup N} \cong Z_M \otimes Z_N$ is natural with respect to $X$-homeomorphisms. This essentially means

(i) Associativity : The composition of identifications obeys the usual associativity constraint. (ii) Naturality : For any $X$-homeomorphisms $\alpha : M \to M^{'}$, $\beta : N \to N^{'}$, the diagram

$$\begin{tikzcd}
Z_{M \sqcup N} \arrow[r, "(\alpha \sqcup \beta)_{\#}"] \arrow[d, "\cong"'] & Z_{M' \sqcup N'} \arrow[d, "\cong"] \\
Z_M \otimes Z_N \arrow[r, "\alpha_{\#} \otimes \beta_{\#}"] & Z_{M'} \otimes Z_{N'}
\end{tikzcd}$$

commutes. These definitions have been discussed by Turaev in [Tur10b] while defining axioms for a TQFT.

(vi) Turaev does not say explicitly about symmetric structure. Given any two $n$-dimensional $X$-manifolds $M$ and $N$, we have an $X$-homeomorphism $\alpha : M \sqcup N \to N \sqcup M$. Correspondingly, we have an isomorphism $\alpha_{\#} : Z_{M \sqcup N} \to Z_{N \sqcup M}$ in $\mathcal{C}$. Then Axiom (ii) gives the isomorphism

$$Z_M \otimes Z_N \cong Z_N \otimes Z_M$$

in $\mathcal{C}$. This forces a *symmetric* structure on the objects $Z_M$ in $\mathcal{C}$. The axioms for symmetricity would follow automatically owing to the natural $X$-homemorphisms between the $n$-manifolds. Suppose the monoidal category $\mathcal{C}$ is already symmetric with $c$ as its braiding. To avoid confusion between the two symmetric structures in $\mathcal{C}$, we introduce a *symmetric* $(n+1)$-dimensional $\mathcal{X}$-HQFT as a $(n+1)$-dimensional $\mathcal{X}$-HQFT in the above sense together with an additional axiom for the braiding $c$ of the category $\mathcal{C}$. Let us say it explicitly:

**Axiom (ix)** For any $n$-dimensional $\mathcal{X}$-manifolds $M$, $N$, the following diagram

$$\begin{tikzcd}
Z_{M \sqcup N} \arrow[r, "\cong"] \arrow[d, "\alpha_{\#}"'] & Z_M \otimes Z_N \arrow[d, "c_{M,N}"] \\
Z_{N \sqcup M} \arrow[r, "\cong"] & Z_N \otimes Z_M
\end{tikzcd}$$

commutes.

Thus the original braiding $c$ of the category $\mathcal{C}$ agrees with the forced symmetric structure on the objects $Z_M$, corresponding to $n$-dimensional $X$-manifolds $M$.

(vii) Rigidity on the objects $Z_M$ in $\mathcal{C}$ is enforced automatically by $X$-HQFT. We set

$$(Z_M)^* = Z_{-M}.$$

The morphisms for the dual pair are the morphisms in $\mathcal{C}$ corresponding to the $X$-cobordisms : $\{\emptyset \to M \sqcup -M\}$ and $\{M \sqcup -M \to \emptyset\}$.

# Chapter 3

# Crossed Systems

In this chapter we introduce the crossed algebras discovered by Turaev. We discuss Homotopy Quantum Field Theory(HQFT) in dimension 2. Finally, we give examples of crossed algebras in a category with twisted associativity. Throughout this chapter, let $G$ be a multiplicative group with a unit $e$, $\mathbb{K}$ a ground field.

## 3.1 G-coalgebras

A group-coalgebra [Vir02] (or simply a $G$-coalgebra when the group is known) is a datum $(\{C_g\}, \{\Delta_{h,g}\}, \epsilon)$ with $h, g \in G$ satisfying the following axioms for all $f, g, h \in G$:

(1) $C_g$ is a vector space,

(2) $\Delta_{f,g} : C_{fg} \to C_f \otimes C_g, \epsilon : C_e \to \mathbb{K}$ are linear maps,

(3) $(\mathrm{id} \otimes \Delta_{g,h}) \circ \Delta_{f,gh} = (\Delta_{f,g} \otimes \mathrm{id}) \circ \Delta_{fg,h} : C_{fgh} \to C_f \otimes C_g \otimes C_h$,

(4) $(\mathrm{id} \otimes \epsilon) \circ \Delta_{g,e} = \mathrm{id} : A_g \to A_g$,

(5) $(\epsilon \otimes \mathrm{id}) \circ \Delta_{e,g} = \mathrm{id} : A_g \to A_g$.

Any $G$-graded coalgebra gives rise to a $G$-coalgebra by considering the graded components of the coalgebra and its structure maps. The opposite is true only for finite groups since if one defines $C = \oplus C_g$, the comultiplication $\Delta = \oplus_{f,g} \Delta_{f,g}$ is not necessarily well-defined. Essentially, group coalgebras are "local" versions of graded coalgebras. The group coalgebras admit two important generalisations. First, one can talk about $G$-coalgebras in any monoidal category. Second, one can talk about $G$-coalgebras for non-discrete group or group schemes which we do in the next chapter.

Let us consider a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, I, \alpha, \lambda, \rho)$. A *G-coalgebra* in this category is a collection of objects $\{C_g\}_{g \in G}$ equipped with the following collection

$$\Delta_{f,g} : C_{fg} \to C_f \otimes C_g$$

and

$$\epsilon : C_e \to I$$

of morphisms in $\mathcal{C}$ for $f, g \in G$ such that the following diagrams commute :

$$\begin{tikzcd}
C_{fgh} \arrow[r, "\Delta_{f,gh}"] \arrow[d, "\Delta_{fg,h}"'] & C_f \otimes C_{gh} \arrow[r, "id\otimes\Delta_{g,h}"] & C_f \otimes (C_g \otimes C_h) \\
C_{fg} \otimes C_h \arrow[r, "\Delta_{f,g}\otimes id"] & (C_f \otimes C_g) \otimes C_h \arrow[ur, "\alpha_{C_f,C_g,C_h}"] &
\end{tikzcd}$$

and,

$$\begin{tikzcd}
C_{g.e} \arrow[r, "\Delta_{g,e}"] & C_g \otimes C_e \arrow[d, "id\otimes\epsilon"] \\
& C_g \otimes I \arrow[ul, "\rho_{C_g}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
C_{e.g} \arrow[r, "\Delta_{e,g}"] & C_e \otimes C_g \arrow[d, "\epsilon\otimes id"] \\
& C_g \otimes I \arrow[ul, "\lambda_{C_g}"]
\end{tikzcd}$$

Along the same lines we define a $G$-algebra in a monoidal category. A *G-algebra* in $\mathcal{C}$ is a collection of objects $A = \{A_g\}_{g \in G}$ equipped with a following collection of morphisms in $\mathcal{C}$ :

$$\mu_{f,g} : A_f \otimes A_g \to A_{fg}$$

and,

$$\eta : I \to A_e$$

for $f, g \in G$ such that they make the following diagrams commute :

$$\begin{tikzcd}
A_f \otimes (A_g \otimes A_h) \arrow[r, "id_f \otimes \mu_{g,h}"] \arrow[d, "\alpha_{f,g,h}"'] & A_f \otimes A_{gh} \arrow[r, "\mu_{f,gh}"] & A_{fgh} \\
(A_f \otimes A_g) \otimes A_h \arrow[r, "\mu_{f,g} \otimes id_h"] & (A_{fg} \otimes A_h) \arrow[ur, "\mu_{fg,h}"] &
\end{tikzcd}$$

and,

$$\begin{tikzcd}
A_g \otimes A_e \arrow[r, "\mu_{g,e}"] & A_g \\
& A_g \otimes I \arrow[ul, "id \otimes \eta"] \arrow[u, "\rho_{A_g}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A_e \otimes A_g \arrow[r, "\mu_{e,g}"] & A_g \\
& I \otimes A_g \arrow[ul, "\eta \otimes id"] \arrow[u, "\lambda_{A_g}"']
\end{tikzcd}$$

In both cases, for all $g \in G$, we say $C_g$(resp, $A_g$) is a component of $C$(resp, $A$). We say a $G$-coalgebra $C = \{C_g\}$ is rigid in $\mathcal{C}$ if every component of C has a dual in $\mathcal{C}$. Given that a group coalgebra $C$ is rigid in $\mathcal{C}$, let us discuss its dual. The dual $C^\sharp$ of $C$ is the collection of objects $\{C_g^*\}_{g \in G}$. It is coming equipped with a collection of multiplications

$$\mu_{g,h} : C_g^* \otimes C_h^* \to (C_h \otimes C_g)^* \xrightarrow{\Delta_{h,g}^*} C_{hg}^*$$

that turn $C^\sharp$ into $G^{op}$-algebra $(\{C_g^*\}, \{\mu_{g,h}\}, \eta)$ with $g, h \in G^{op}$ and where $G^{op}$ is the group $(G, *^{op})$ with the opposite multiplication given as $g *^{op} h = hg$ and $\eta$ is given by $\epsilon^*$. For the second arrow in the above diagram we use the fact that taking a dual is a contravariant functor $* : \mathcal{C} \to \mathcal{C}$. In fact the dual $C^\sharp$ forms a $G^{op}$-algebra in $\mathcal{C}$.

Similarly we have the dual of a $G$-algebra $A$. Its defined as the collection $A^* = \{(A_{g^{-1}})^* | g \in G\}$ of objects in $\mathcal{C}$. We set a notation here which we carry throughout. For all objects in $\mathcal{C}$ let us set $A_{g^{-1}}^* = (A_{g^{-1}})^*$ and $\mu_{f,g}^* = (\mu_{f,g})^*$ for the morphisms in $\mathcal{C}$, where $f, g$ is in $G$. The conjugate maps $\mu_{g^{-1},f^{-1}}^* : A_{(fg)^{-1}}^* \to A_{f^{-1}}^* \otimes A_{g^{-1}}^*$ and $\eta^* : A_e^* \to I$ for $f, g \in G$, makes $A^*$ a $G$-coalgebra. We would write this collection as $A^* = (A^*, \Delta, \epsilon)$, where $\Delta = \{\Delta_{f,g} = \mu_{g^{-1},f^{-1}}^* | f, g \in G\}$ and $\epsilon = \eta^*$.

In the category $Vect_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$, there are two notions of a dual: *the inner dual* and *the outer dual* [Zun04a]. The outer dual $C^\sharp$ is simply the same as the dual in a monoidal category as defined above. The direct sum $\oplus_{g \in G} C_g^*$ becomes a $G^{op}$-graded algebra. The inner dual of $C$ is just a vector space $(\oplus_{g \in G} C_g^*) \otimes \mathbb{K}G$ but this will play a role only later in Chapter 4, Section (4.4.1). Note that in case $G = \{1\}$, $G$-algebras and $G$-coalgebras are nothing but monoids and comonoids in $\mathcal{C}$ respectively .

## 3.2 Frobenius graded systems

Recently work has been done to establish an interesting connection between the notion of Frobenius algebra or the more general Frobenius extension on the one hand and 2-dimensional topological quantum field theories on the other hand. The observation that 2-dimensional TQFTs are essentially the same thing as commutative Frobenius algebras was first made by R.Dijkgraaf in his Ph.D. thesis. More precise proofs have been given by Quinn, Swain, and Abrams. There are interesting possibilities for further interactions with the theories of Atiyah, Drinfeld, Jones, Turaev and Witten.

In this section, we develop the theory of Frobenius extensions in a monoidal category. The three equivalent characterisations of Frobenius extensions in such a category are discussed in the form of a small result. We then go on further to define a Frobenius graded system. A similar characterisation in the graded case is also analysed.

### 3.2.1 Frobenius systems

Let $(R, \mu, \eta)$ be a monoid in a symmetric monoidal category $(\mathcal{C}, \otimes, I, a, \lambda, \gamma, \tau)$. Let us assume that $(R, R^*)$ is a dual pair in $\mathcal{C}$ with $u_R : I \to R \otimes R^*$ and $v_R : R^* \otimes R \to I$. Note that $(R^*, \mu^*, \eta^*)$ forms a comonoid in $\mathcal{C}$. In the category $R$-Mod of left $R$ modules, objects are pairs $(M, \alpha_M)$ where $M$ is an object in $\mathcal{C}$ together with a left $R$-action $\alpha_M$ as a morphism in $\mathcal{C}$. The morphism $\alpha_M : R \otimes M \to M$ is such that it makes the following diagrams commute

$$\begin{tikzcd}
R \otimes R \otimes M \arrow[r, "Id_R \otimes \alpha_M"] \arrow[d, "\mu \otimes Id_M"'] & R \otimes M \arrow[d, "\alpha_M"] \\
R \otimes M \arrow[r, "\alpha_M"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes M \arrow[r, "\eta \otimes Id_M"] \arrow[dr, "\lambda_M"'] & R \otimes M \arrow[d, "\alpha_M"] \\
 & M
\end{tikzcd}$$

A morphism $f : M \to N$ in $R$-Mod is a morphism in $\mathcal{C}$ which makes the following diagram of left $R$-action commute.

$$\begin{tikzcd}
R \otimes M \arrow[r, "\alpha_M"] \arrow[d, "Id_R \otimes f"'] & M \arrow[d, "f"] \\
R \otimes N \arrow[r, "\alpha_N"] & N
\end{tikzcd}$$

Similarly one can define right $R$-action and construct the category Mod-$R$ of right $R$ modules. Note that any left $R$ module, say $(M, \alpha_M)$, automatically gets a left $R^*$-coaction $\widehat{\alpha}_M$ given by

$$\widehat{\alpha}_M : M \xrightarrow{\cong} I \otimes M \to R^* \otimes R \otimes M \xrightarrow{id_{R^*} \otimes \alpha_M} R^* \otimes M,$$

which satisfies following conditions:

$$(\mu^* \otimes id) \circ \widehat{\alpha}_M = (id \otimes \widehat{\alpha}_M)\widehat{\alpha}_M, \tag{3.1}$$

$$(\eta^* \otimes id) \circ \widehat{\alpha}_M = id. \tag{3.2}$$

Similarly an $R$-module has a right $R^*$-coaction. The forgetful functor For : $R$-Mod $\to$ $\mathcal{C}$ assigns to every $R$-module $M$, an object $M_{\mathcal{C}}$ in $\mathcal{C}$ which is the same on the level of objects but now we forget about the $R$-action on $M$. For any object $M$ in $\mathcal{C}$, $R \otimes M$ is an object in $\mathcal{C}$. However, $R \otimes M$ also has a structure of a left $R$ module defined by $\alpha_{R\otimes M} = \mu \otimes Id_M$. Similarly $M \otimes R$ is an object in $\mathcal{C}$ together with a right $R$ module structure defined by $\alpha_{N\otimes R} = Id_N \otimes \mu$. The induction functor Ind : $\mathcal{C} \to R$-Mod is defined by Ind$(M) = R \otimes M$. Using the fact that $R$ is a rigid object in $\mathcal{C}$, we can formalise the coinduction functor as follows. CoInd : $\mathcal{C} \to R$-Mod is defined as CoInd$(M) = hom(R, M)$. Here $hom$ is the internal $Hom$, an object in $\mathcal{C}$, which is defined as $hom(R, M) = R^* \otimes M$. There is a left $R$-action on internal $Hom$ which is given by the composition of the following maps:

$$R \otimes R^* \otimes M \xrightarrow{Id_R \otimes \mu^* \otimes Id_M} R \otimes R^* \otimes R^* \otimes M \xrightarrow{\tau \otimes Id_{R^*} \otimes M} R^* \otimes R \otimes R^* \otimes M \xrightarrow{v_R \otimes Id_R^* \otimes Id_M} R^* \otimes M.$$

**Proposition 3.2.1** *The induction functor* Ind *defined as above is left adjoint to the for-*

*getful functor.*

**Proposition 3.2.2** *The coinduction functor* CoInd *defined above is right adjoint to the forgetful functor.*

The proofs of the above two propositions are standard. Note that Proposition 3.2.1 can also be interpreted in terms of functors as follows. Suppose $(R, \mu, \eta)$ is a monoid in $\mathcal{C}$. Then the functor $T = R \otimes \_$ is a monad in $\mathcal{C}$. The monad structure $(T, m, i)$ with natural transformations $m : TT \to T$ and $i : I_{\mathcal{C}} \to T$ is defined as follows. The former transformation uses the multiplication of $R$ and the unit of $R$ defines the later. Explicitly,

$$m_A : R \otimes R \otimes A \xrightarrow{\mu \otimes 1_A} R \otimes A,$$

$$i_A : A \xrightarrow{\gamma} I_{\mathcal{C}} \otimes A \xrightarrow{\eta \otimes 1_A} R \otimes A$$

for any $A \in \mathrm{Ob}\, \mathcal{C}$.

Now, given a monad $T$ in $\mathcal{C}$, $\langle TA, m_A \rangle$ is a $T$-algebra. In fact a $T$-algebra is simply a left $R$-module. Moreover, $R$-Mod is the category of $T$-algebras, which in literature is denoted by $\mathcal{C}^T$. Then by Theorem 2.1.2, a monad $T$ in $\mathcal{C}$ defines an adjunction $< F^T, G^T, \eta^t, \epsilon^T >: \mathcal{C} \to \mathcal{C}^T$, where

$$G^T : \mathcal{C}^T \to \mathcal{C},$$

$$F^T : \mathcal{C} \to \mathcal{C}^T$$

are given as follows. The functor $G^T$ simply forgets the structure map of each $T$-algebra; where as the functor $F^T$ is defined by $A \mapsto \langle TA, m_A \rangle$. Clearly, $G^T$ is the forgetful functor which we denoted by For and $F^T$ is the Ind functor. The Ind functor is precisely the free algebra functor and it is a left adjoint to the forgetful functor as a part of the free algebra-forgetful adjunction (Theorem 2.1.2). Proposition 3.2.2 is dual to 3.2.1 which will form the forgetful $-$ free coalgebra adjunction. See also [BW05].

Before we give the main result of this section, recall the definitions of a dual pairing and a non-degenerate form discussed in Section 2.1.

We are now in a position to give the main idea of this section and to prove the key result. It is interesting to compare the induction and coinduction functors. In particular, we want to see what happens if they are isomorphic.

**Definition 3.2.3** *We say the monoid $(R, \mu, \eta)$ is a $Frobenius$ extension if the associated induction and coinduction functors are naturally isomorphic.*

In particular, in the category of $R$-modules the natural isomorphism essentially means $R \otimes M \cong M \otimes R^*$ for any $R$-module $M$.

**Remark.** Propositions 3.2.1 and 3.2.2 implies respectively that the functors Ind and CoInd are left and right adjoint to For. Then another way of looking at Definition 3.2.4 of a Frobenius extension is that Ind (or CoInd) is at the same time a left and a right adjoint of the forgetful functor. Thus Definition 3.2.4 is equivalent to say that $R \otimes \_$ is a Frobenius functor.

**Proposition 3.2.4** *Let $(R, \mu, \eta)$ be a rigid monoid in a monoidal category $\mathcal{C}$. Then the folowing are equivalent :*

*(i) The dual object $R^*$ of $R$ is a left $R$ module and $R$ is isomorphic to $R^*$ in $R$-*Mod.

*(ii) There exists a morphism $\Delta : R \to R \otimes R$ in $R$-*Mod *and a morphism $\epsilon : R \to I$ in $\mathcal{C}$, such that $(Id \otimes \Delta) \circ \Delta = (\Delta \otimes Id) \circ \Delta$ and $(Id \otimes \epsilon) \circ \Delta = Id = (\epsilon \otimes Id) \circ \Delta$. And, $\epsilon$ gives rise to a non-degenerate pairing :*

$$R \otimes R \xrightarrow{\epsilon.\mu} I.$$

*with :*

$$I \xrightarrow{\Delta.\eta} R \otimes R.$$

*as its copairing.*

*(iii) $(R, \mu, \eta)$ is a Frobenius extension.*

Proof: $(i) \iff (ii)$. Given that $\phi : {}_R R \cong {}_R R^*$, define $\Delta$ as the composition of the following maps:

$$R \xrightarrow{\phi} R^* \xrightarrow{\mu^*} R^* \otimes R^* \xrightarrow{\phi^{-1} \otimes \phi^{-1}} R \otimes R.$$

The composition

$$R \xrightarrow{\phi} R^* \xrightarrow{\eta^*} I$$

gives $\epsilon$. Its easy to check that axioms are satisfied. The non-degeneracy of $\epsilon$ follows from the axioms of dual of $R$. Conversely,

$$R \otimes R \xrightarrow{\mu} R \xrightarrow{\epsilon} I$$

and,

$$I \xrightarrow{\eta} R \xrightarrow{\Delta} R \otimes R$$

together with the non-degeneracy of $\epsilon$ implies that $R \cong R^*$ in $\mathcal{C}$.

$(i) \iff (iii)$. Given that $\phi : {}_R R \cong {}_R R^*$, then automatically, $R \otimes M \cong R^* \otimes M$ holds functorially. Conversely, substituting the unit object into the natural isomorphism of functors gives the required isomorphism in $R$-Mod.

□

**Remark.** In this case we say $(R, \mu, \eta, \Delta, \epsilon)$ forms a *Frobenius system* in $\mathcal{C}$. Note here that a Frobenius system is a Frobenius extension with a fixed natural isomorphism. In the case when we are in the category of vector spaces over a ground field $\mathbb{K}$, a monoid $(R, \mu, \eta)$ is a $\mathbb{K}$-algebra. A Frobenius extension is a Frobenius $\mathbb{K}$-algebra whereas a Frobenius system is a Frobenius $\mathbb{K}$-algebra with a fixed pairing.

**Examples.** It is easy to show that first three examples below are Frobenius systems. The first one was studied by Khovanov [Kho00]. The third one was studied by Lee [Lee05]. In the first three examples, we are considering the category of $I$-modules, and provide a Frobenius system $\mathcal{F} = (R, \Delta, \epsilon)$ in these categories. The last example, discussed by Brzeziński [Brz00], gives a Frobenius extension starting from an entwining structures over a ground field $\mathbb{K}$.

(i) $I = \mathbb{Z}$ and $R = \mathbb{Z}[X] \;/ < X^2 >$

$\Delta(1) = 1 \otimes X + X \otimes 1 \qquad \epsilon(1) = 0$

$\Delta(X) = X \otimes X \qquad \epsilon(X) = 1.$

(ii) $I = \mathbb{Z}[c]$ and $R = \mathbb{Z}[X, c] \;/ < X^2 >$

$\Delta(1) = 1 \otimes X + X \otimes 1 + cZ \otimes X \qquad \epsilon(1) = -c$

$$\Delta(X) = X \otimes X \qquad\qquad \epsilon(X) = 1.$$

(iii) $I = \mathbb{Q}[c]$ and $R = \mathbb{Q}[X]\ /< X^2 - 1 >$

$$\Delta(1) = 1 \otimes X + X \otimes 1 \qquad\qquad \epsilon(1) = 0;$$

$$\Delta(X) = X \otimes X + 1 \otimes 1 \qquad\qquad \epsilon(X) = 1.$$

(iv) [Brz00] Let $(A, C)_\psi$ be an entwining structure over a commutative ring $K$ and let $B = C^{*op}$. If $A$ is a faithfully flat $K$-module and $C$ is a finitely generated projective $K$-module, and let $X = B\sharp_{\overline{\psi}} A$ such that the functor $\mathrm{Hom}(B, \_) : M_A \to M_X$ is the left adjoint of the functor $\mathcal{F} : M_X \to M_A$ induced by $A \hookrightarrow X$, then the extension $A \subseteq X$ is Frobenius. Here, $M_A$ denotes the category of right $A$-modules. If we are considering an entwining structure $(A, C)_\psi$ over a field $\mathbb{K}$, then $A$ is already a faithfully flat and projective $\mathbb{K}$-module (as it is simply a vector space).

### 3.2.2 Frobenius graded systems

Assume $A = \{A_g, \mu_{g,h}, \eta | g, h \in G\}$ is a $G$-algebra. We define the category $A$-Mod with objects as pairs $(M, \alpha_M)$ and call them as left $A$-modules, (or, simply $A$-modules). Here $M = \{M_g | g \in G\}$ is a collection of objects in $\mathcal{C}$ equipped with an $A$-action given by the collection $\alpha_M = \{\alpha_M^{g,h} : A_g \otimes M_h \to M_{gh}\}$ of morphisms in $\mathcal{C}$. These objects and morphisms are such that for any $f, g, h \in G$ the following diagrams commute:

$$\begin{tikzcd} A_f \otimes A_g \otimes M_h \arrow[r, "Id_{A_f} \otimes \alpha_M^{g,h}"] \arrow[d, "\mu_{f,g} \otimes Id_{M_f}"'] & A_f \otimes M_{gh} \arrow[d, "\alpha_M^{f,gh}"] \\ A_{fg} \otimes M_h \arrow[r, "\alpha_M^{fg,h}"'] & M_{fgh} \end{tikzcd} \qquad \begin{tikzcd} I \otimes M_f \arrow[r, "\eta \otimes Id_{M_f}"] \arrow[dr, "\lambda_{M_f}"'] & A_e \otimes M_f \arrow[d, "\alpha_M^{e,f}"] \\ & M_f. \end{tikzcd}$$

A morphism $\phi$ between two objects $(M, \alpha_M)$ and $(N, \alpha_N)$ in $A$-Mod is a collection of morphisms in $\mathcal{C}$ given by $\phi = \{\phi_g : M_g \to N_g | g \in G\}$ such that it commutes with the

$A$-action of the two objects. Precisely,

$$\begin{tikzcd}
A_f \otimes M_g \arrow[r, "Id_{A_f} \otimes \phi_g"] \arrow[d, "\alpha_M^{f,g}"'] & A_f \otimes N_g \arrow[d, "\alpha_N^{f,g}"] \\
A_{fg} \otimes M_h \arrow[r, "\alpha_M^{fg,h}"'] & M_{fgh}
\end{tikzcd}$$

commutes.

In particular, $A$ itself can be considered as an $A$-module with left $A$-action given by $\mu$. Likewise one defines right $A$-action for Mod-$A$ category. Note that if $A$ is rigid in $\mathcal{C}$ then any left $A$-module, say $(M, \alpha_M)$ , automatically gets a left $A^*$-coaction $\widehat{\alpha}_M$ given by

$$\widehat{\alpha}_M^{g,h} : M_{gh} \xrightarrow{\cong} I \otimes M_{gh} \to A_g^* \otimes A_{g^{-1}} \otimes M_{gh} \xrightarrow{id \otimes \alpha_M^{g,h}} A_g^* \otimes M_h,$$

which satisfies following conditions:

$$(\mu^* \otimes id) \circ \widehat{\alpha}_M = (id \otimes \widehat{\alpha}_M)\widehat{\alpha}_M, \tag{3.3}$$

$$(\eta^* \otimes id) \circ \widehat{\alpha}_M = id. \tag{3.4}$$

We can define the category $A_e$-Mod of left $A_e$ modules exactly as in the previous subsection. Note that for any object $M$ in $\mathcal{C}$, $A_e \otimes M$ is a left $A_e$ module with action given by $A_e \otimes (A_e \otimes M) \xrightarrow{\mu \otimes 1} A_e \otimes M$.

Analogously to the forgetful functor defined in the last subsection, we have a restriction functor $\mathrm{Res} : A\text{-Mod} \to \mathcal{C}$ defined as $\mathrm{Res}((M_g, \alpha_M)) = M_e$.

Define the induction functor $\mathrm{Ind} : \mathcal{C} \longrightarrow A - \mathrm{Mod}$ as follows:

$$\mathrm{Ind}(M)_g \quad = \quad A_g \otimes M.$$

It is easy to see that $\mathrm{Ind}(M)_g$ is an object in $A$-Mod. Next, let us assume that the $G$-algebra $A$ is rigid in $\mathcal{C}$. Then $X = \{X_g^* = A_{g^{-1}}^* \otimes M | g \in G\}$ is also in $A$-Mod. For

$f, g \in G$, the $A$-action $\alpha_X^{f,g} : A_f \otimes X_g \to X_{fg}$ on $X$ is given as :

$$\begin{tikzcd}
A_f \otimes X_g = A_f \otimes A^*_{g^{-1}} \otimes M \arrow[r, "Id_{A_f} \otimes \Delta_{f,g^{-1}f^{-1}} \otimes Id_M"] & A_f \otimes A^*_f \otimes A^*_{g^{-1}f^{-1}} \otimes M \arrow[d, "\tau \otimes Id_{A_{g^{-1}f^{-1}}} \otimes Id_M"] \\
& A^*_f \otimes A_f \otimes A^*_{g^{-1}f^{-1}} \otimes M \arrow[d, "v_{A_f} \otimes Id_M"] \\
& A^*_{g^{-1}f^{-1}} \otimes M = A^*_{(fg)^{-1}} \otimes M = X_{fg}.
\end{tikzcd}$$

**Proposition 3.2.5** *If $A$ is a $G$-algebra, then its dual $A^*$ is a $G$-coalgebra which is an $A$-module.*

Proof: We have seen that $A^*$ is a $G$-graded algebra. In particular, choosing $M$ as the unit of $\mathcal{C}$ in the above diagram, we conclude that $A^*$ is an $A$-module. $\square$

Now define coinduction functor $\mathrm{CoInd} : \mathcal{C} \to A - \mathrm{Mod}$ as :

$$\mathrm{CoInd}(M)_g \quad = \quad A^*_{g^{-1}} \otimes M.$$

Analogous to Proposition 3.2.1 and 3.2.2, we have the following result :

**Proposition 3.2.6** *The induction functor is left adjoint and the coinduction functor is right adjoint to the restriction functor.*

Proof: Define the following maps :

$$\chi_{M,N} : \mathrm{Hom}_{\mathcal{C}}(M, \mathrm{For}N) \longrightarrow \mathrm{Hom}_{A-\mathrm{Mod}}(\mathrm{Ind}M, N)$$

and,

$$\rho_{M,N} : \mathrm{Hom}_{A-\mathrm{Mod}}(\mathrm{Ind}M, N) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(M, \mathrm{For}N)$$

as the composition of :

$$(\chi_{M,N}(\phi))_g : A_g \otimes M \xrightarrow{id \otimes \phi} A_g \otimes N_e \xrightarrow{\alpha_N^{g,e}} N_g$$

and,

$$\rho_{M,N}(\psi) : M \xrightarrow{\lambda_M^{-1}} I \otimes M \xrightarrow{\eta \otimes id} A_e \otimes M \xrightarrow{\psi} N_e$$

respectively, where $\phi \in \mathrm{Hom}_{\mathcal{C}}(M, N_e)$ and $\psi \in \mathrm{Hom}_{A-\mathrm{Mod}}((A_g \otimes M), N)$. Now repeating the proof of Proposition 3.2.1 with the new definition of $\chi_{M,N}$ we can similarly show that the induction functor is left adjoint to the restriction functor. For the second part of the proposition, define the following maps :

$$\rho_{M,N} : \mathrm{Hom}_{A-\mathrm{Mod}}(N, \mathrm{CoInd}(M)) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathrm{Res}N, M)$$

and,

$$\chi_{M,N} : \mathrm{Hom}_{\mathcal{C}}(\mathrm{Res}N, M) \longrightarrow \mathrm{Hom}_{A-\mathrm{Mod}}(N, \mathrm{CoInd}(M))$$

as the composition of :

$$\rho_{M,N}(\phi) : N_e \xrightarrow{\phi_e} A_e^* \otimes M \xrightarrow{\eta^* \otimes id} I \otimes M \xrightarrow{\lambda_M} M.$$

and,

$$(\chi_{M,N}(\psi))_g : N_g \xrightarrow{\widehat{\alpha}_N^{g,e}} A_g^* \otimes N_e \xrightarrow{id \otimes \psi} A_g^* \otimes M,$$

respectively, where $\phi \in \mathrm{Hom}_{A-\mathrm{Mod}}(N, (A_{g^{-1}}^* \otimes M))$ and $\psi \in \mathrm{Hom}_{\mathcal{C}}(N_e, M)$. Repeat the proof of Proposition 3.2.2. $\square$

Next thing we want to do is to compare the induction and coinduction functors. We give the following definition before the main result of the section.

**Definition 3.2.7** *A Frobenius G-algebra $(A, \mu, \eta)$ is a rigid G-algebra in a monoidal category $\mathcal{C}$ such that the associated induction and coinduction functors are naturally isomorphic.*

Note that in the category of $\mathbb{K}$-vector spaces, the natural isomorphism would imply that for each $g \in G$, $A_g \otimes M \cong A_{g^{-1}}^* \otimes M$ where $A = \bigoplus_{g \in G} A_g$ is a $G$-graded algebra and $M$ is a vector space over $\mathbb{K}$.

Observe that the dual of a $G$-algebra $A$ is a $G$-coalgebra $A^*$ which is essentially $\mathrm{CoInd}(I)$. It has an $A$-module structure, by Proposition 3.2.5. We have the following result:

**Theorem 3.2.8** *Let $(A, \mu, \eta)$ be a rigid $G$-algebra in a symmetric monoidal category $\mathcal{C}$. Then $\eta$ is a Frobenius $G$-algebra if and only if one of the following statements hold.*

*(i) $A$ is isomorphic to its dual in $A$-*Mod.

*(ii) There exists a collection of morphisms, $\Delta = \{\Delta_{g,h} : A_{gh} \to A_g \otimes A_h\}$ in $A$-*Mod *called comultiplications and a morphism $\epsilon : A_e \to I$ in $\mathcal{C}$ called counit, such that $(Id_{A_f} \otimes \Delta_{g,h}) \circ \Delta_{f,gh} = (\Delta_{f,g} \otimes Id_{A_h}) \circ \Delta_{f,gh}$ and $(Id_{A_f} \otimes \epsilon) \circ \Delta_{f,e} = Id_{A_f} = (\epsilon \otimes Id_{A_f}) \circ \Delta_{e,f}$. Then $\epsilon$ and $\Delta$ give rise to non-degenerate pairings $\zeta = \{\zeta_g\}_{g\in G}$:*

$$\zeta_g : A_g \otimes A_{g^{-1}} \xrightarrow{\epsilon.\mu_{g,g^{-1}}} I, \tag{3.5}$$

*with copairings $\widetilde{\zeta} = \{\widetilde{\zeta}_g\}_{g\in G}$:*

$$\widetilde{\zeta}_g : I \xrightarrow{\Delta_{g,g^{-1}}.\eta} A_g \otimes A_{g^{-1}} \tag{3.6}$$

*and conversely, the set of pairings $\{\widetilde{\zeta}_g\}$ give rise to comultiplications and the counit.*

*(iii) $A$ is Frobenius $G$-algebra.*

Proof: $(i) \iff (ii)$. Let $\phi : {}_AA \cong {}_AA^*$ be the given isomorphism. This means we have a collection of isomorphisms, $\phi_g : A_g \to A^*_{g^{-1}}$ in $\mathcal{C}$, and that these morphisms preserve the left $A$ structure.

Define $\Delta_{g,h}$ as the composition of the following maps:

$$A_{gh} \xrightarrow{\phi_{gh}} A^*_{(gh)^{-1}} \xrightarrow{\mu^*_{h^{-1},g^{-1}}} A^*_{g^{-1}} \otimes A^*_{h^{-1}} \xrightarrow{\phi_g^{-1}\otimes\phi_h^{-1}} A_g \otimes A_h.$$

The composition

$$A_e \xrightarrow{\phi_e} A^*_e \xrightarrow{\eta^*} I$$

gives $\epsilon$. The axioms follow easily using the duality of $A$ and $A^*$. Once we have the comultiplications and the counit, we can define pairings and copairings and conversely. Equations (3.5) and (3.6) define pairings/copairings. Then non-degeneracy of the form follows from the axioms of $\Delta$ and $\epsilon$. On the other hand, given the non-degenerate pairings, $\zeta_g : A_g \otimes A_{g^{-1}} \to I$ and their copairings $\widetilde{\zeta}_g : I \to A_g \otimes A_{g^{-1}}$, we define comultiplications and counit as:

$$\Delta_{g,h} : A_{gh} \cong A_{gh} \otimes I \xrightarrow{1\otimes\widetilde{\zeta}_{h^{-1}}} A_{gh} \otimes A_{h^{-1}} \otimes A_h \xrightarrow{\mu_{gh,h^{-1}}\otimes 1} A_g \otimes A_h \text{ ; and}$$

$$\epsilon : A_1 \xrightarrow{\Delta_{1,1^{-1}}} A_1 \otimes A_{1^{-1}} \xrightarrow{\zeta_1} I,$$

and the axioms would follow from non-degeneracy of the form.

Conversely let condition (ii) be given. Observe that for each $g \in G$, $A^*_{g^{-1}} = A_g \cong A_g$ in $\mathcal{C}$. Hence $A^* \cong A$ as $G$-algebras in $\mathcal{C}$. Moreover, both the $G$-algebras $A$ and $A^*$ are $A$-modules (by Proposition 3.2.5), so they are isomorphic in $A$-Mod as well.

$(i) \iff (iii)$. Given that $\phi : {}_AA \cong {}_AA^*$, then automatically, $A \otimes M \cong A^* \otimes M$ holds functorially. Conversely, substituting the unit object into the natural isomorphism of functors gives the required isomorphism in $A$-Mod. $\square$

**Definition 3.2.9** *A Frobenius G-graded system is an ordered pentuple* $\mathcal{F} = (A, \mu, \eta, \Delta, \epsilon)$ *where* $(A, \mu, \eta)$ *is a Frobenius G-algebra and* $\Delta$ *and* $\epsilon$ *are chosen in such a way that they satisfy the Theorem 3.2.8.*

Again, the Frobenius graded system is a Frobenius $G$-algebra together with a fixed natural isomorphism between induction and coinduction functors. Note that when $G = \{1\}$, it is nothing but a Frobenius system in non-graded case. In the case when we are in the category of vector spaces over a ground field $\mathbb{K}$, a $G$-algebra $A$ is a $G$-graded $\mathbb{K}$-algebra, $A = \bigoplus_{g \in G} A_g$ and a Frobenius system is a graded Frobenius algebra with a fixed homogeneous pairing.

## 3.3 Cobordism category

Let $\mathcal{X} = (X, x)$ be a pointed path-connected topological space. We are going to define a symmetric monoidal category $\mathcal{X} - \mathrm{Cob}_n$ in degree $n$, which is crucial for defining an HQFT in a categorical set-up. In the case when $X$ is a point these become the standard definitions used in TQFT. The number $n$ is the dimension of the topological manifolds considered as objects.

We start by defining a weak 2-category $\mathcal{X}-\widetilde{\mathrm{Cob}}_n$. It is weak in two senses. First, the associativity and the identity properties of compositions of 1-morphisms holds only up to a 2-isomorphism. Second, the composition of 2-morphisms is not associative either, although one could make it associative up to a 3-morphism by turning $\mathcal{X}-\widetilde{\mathrm{Cob}}_n$ into a 3-category. The weak 2-category $\mathcal{X}-\widetilde{\mathrm{Cob}}_n$ plays an auxiliary role and its exact axioms are of no significance for the further discussion.

We want to avoid set-theoretical differences, so we draw our manifolds from a sufficiently large universum. By a manifold we understand a compact oriented topological manifold with boundary. A closed manifold would mean a manifold in the above sense but now without boundary. *Sufficiently large* means that each manifold will have a homeomorphic manifold in the universum. The dimension of a manifold is the dimension of any of its components that must be equal for the dimension to exist.

Let us start by describing objects (0-morphisms) of $\mathcal{X}-\widetilde{\mathrm{Cob}}_n$. An object is a triple $\mathcal{M}=(M,f_M,p_M)$ where $M$, called the *base space* of $\mathcal{M}$, is a closed manifold of dimension $n$, such that every component of $M$ is a pointed closed oriented manifold, $f_M : M \to X$ is a continuous function and $p_M$ is a point on each component of $M$. It is convenient to think of $p$ as a function $p_M : \pi_0(M) \to M$ such that $p_M(X) \in X$ for any component $X$. The continuous function $f_M$, called as the *characteristic map* of $M$, is required to be a morphism of pointed manifolds, that is, $f_M(p_M(X)) = x$ for any $X \in \pi_0(M)$. That is to say it sends the base points of all the components of $M$ into $x$. We sometimes may also refer $\mathcal{M}$ as an $\mathcal{X}$-manifold. We think of the empty set as an $n$-dimensional manifold for any $n$. There are unique functions $f : \emptyset \to X$ and $p : \emptyset \to \emptyset$ that turn the empty set into an object which we also denote $\emptyset$.

An $\mathcal{X}$-*homeomorphism* of 0-morphisms $\Psi : \mathcal{M} \to \mathcal{K}$ is an orientation preserving homeomorphism from $M$ to $K$, sending base points of $M$ onto those of $K$ such that $f_M = f_K\Psi$ where $f_M, f_K$ are the characteristic maps of $\mathcal{M}$, $\mathcal{K}$ respectively.

Before we describe 1-morphisms let us point out a couple of operations. We define the disjoint union of 0-morphisms by $(L,f_L,p_L)\sqcup(M,f_M,p_M)=(L\sqcup M, f_L\sqcup f_M, p_L\sqcup p_M)$. Note that if $M, N$ are pointed topological manifolds, then $M\sqcup N$ is again a pointed topological manifold. Moreover, if they are oriented, then we can define a unique orientation on the disjoint union. The orientation on $M \sqcup N$ is such that the inclusion maps are orientation-preserving. The map $f_L \sqcup f_M : L \sqcup M \to X$ is defined in the obvious way, as discussed in Section (2.7). It is a continuous map and let us denote it by $f_{L \sqcup M}$. Just as for $f_L$, we define $p_{L \sqcup M} = p_L \sqcup p_M : \pi_0(L \sqcup M) \to L \sqcup M$. Note that $\pi_0(L \sqcup M) = \pi_0 L \sqcup \pi_0 M$. Indeed, $\mathcal{L} \sqcup \mathcal{M} = (L \sqcup M, f_{L \sqcup M}, p_{L \sqcup M})$ is an $\mathcal{X}$-manifold and hence again a 0-morphism. With the operation of disjoint union we wish to make $\mathcal{X} - \mathrm{Cob}_n$ into a symmetric monoidal category. For shorthand we would simply write $\mathcal{L} \sqcup \mathcal{M}$. Further, for any manifold $M$, let $-M$ be the same manifold with the opposite orientation. For $\mathcal{M} = (M, f, p)$, define $\mathcal{M}^* = (-M, f, p)$. Clearly, $\mathcal{M}^*$ is again a 0-morphism.

Now 1-morphisms in $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$ are cobordisms over $\mathcal{X}$. More precisely, a morphism from $\mathcal{M} = (M, f_M, p_M)$ to $\mathcal{K} = (K, f_K, p_K)$ is a triple $\mathcal{A} = (A, f_A, \alpha_A)$ where $A$, called the *base space* of $\mathcal{A}$, is a manifold of dimension $n+1$, $f_A : A \to X$ is a continuous map, called characteristic map of $A$, and $\alpha_A : \partial A \to (-M) \sqcup K$, called the *boundary map* of $A$, is an $\mathcal{X}$-homeomorphism. By 0-morphism related to $\mathcal{A}$, we understand the 0-morphisms $\mathcal{M}$ and $\mathcal{K}$. $A$ also has a canonical map $p_A : \pi_0 : (\delta A) \to A$ referred to as pointed structure on the boundary of $A$. We call $\partial A_0 = \alpha_A^{-1}(-M)$ as the in-boundary and $\partial A_1 = \alpha_A^{-1}(K)$ as out-boundary for $A$. So, the boundary map is $\alpha_A = \alpha_A^0 \sqcup \alpha_A^1 : \partial A_0 \sqcup \partial A_1 \to -M \sqcup K$.

The composition $\mathcal{B} \circ \mathcal{A}$ of two morphisms $\mathcal{A} : \mathcal{K} \to \mathcal{L}$ and $\mathcal{B} : \mathcal{L} \to \mathcal{M}$ is gluing over the common boundary $L$. The manifold of the composition is $A \sqcup B/\sim$ where the only nontrivial equivalences $a \sim b$ occur when $a \in \partial A$, $b \in \partial B$ and $\alpha_A(a) = \alpha_B(b) \in L$. There is an advantage of working in the topological category here as the manifold structure is uniquely defined on the gluing. Since $L$ induces opposite orientation from $\mathcal{A}$ and $\mathcal{B}$ respectively, thus the orientations of $\mathcal{A}$ and $\mathcal{B}$ are extended/continued on their gluing $\mathcal{B} \circ \mathcal{A}$, see Section (2.8), Theorem (2.8.4). The characteristic map of $\mathcal{A} \sqcup \mathcal{B}$ is $f_A \sqcup f_B/\sim$. Assume $\partial(A \sqcup B/\sim) = X \sqcup Y$, where $X = \alpha_A^{-1}(-K) \subseteq \partial A$ and $Y = \alpha_B^{-1}(M) \subseteq \partial B$, then the boundary identification on the gluing is given by $\alpha_{\{A \sqcup B/\sim\}} = \alpha_A|_X \sqcup \alpha_B|_Y$.

Notice that $(\mathcal{C} \circ \mathcal{B}) \circ \mathcal{A}$ is not equal to $\mathcal{C} \circ (\mathcal{B} \circ \mathcal{A})$ but there is a canonical homeomorphism between the underlying cobordisms preserving the additional data. We will come back to it when we discuss the 2-morphisms.

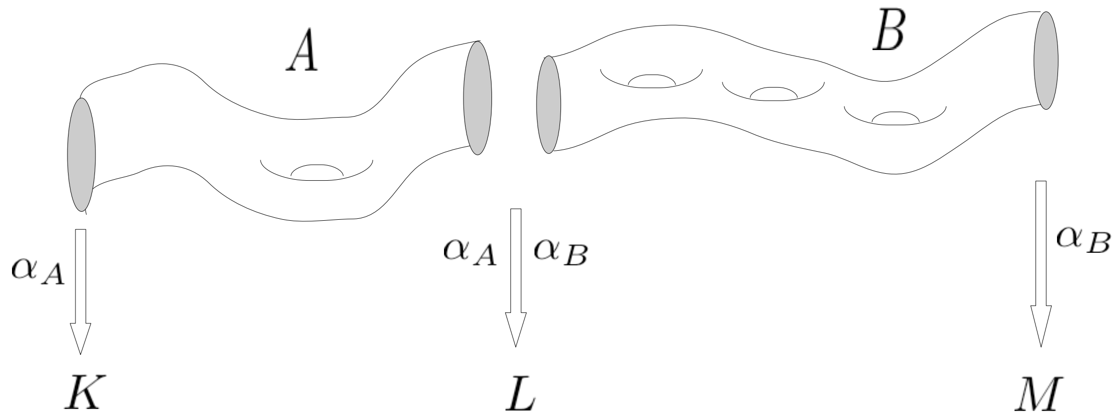



Figure 3.1:

We define the identity morphism $I_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ as the cylinder $M \times [0,1]$ with the function $f(a,t) = f_M(a)$ and the identity map on each part of the boundary. It is not identity in the conventional sense as $\mathcal{A}$ and $I_{\mathcal{M}} \circ \mathcal{A}$ are different even though homeomorphic. But note that in general there is no canonical homeomorphism between them.

An *$\mathcal{X}$-homeomorphism* **of 1-morphisms** $\Phi : \mathcal{A} \to \mathcal{B}$ is an orientation preserving homeomorphism from $A$ to $B$, sending base points of $A$ onto those of $B$ such that $f_A = f_B\Phi$ where $f_A, f_B$ are the characteristic maps of $\mathcal{A}$, $\mathcal{B}$ respectively. Further, we also require the boundary maps to be preserved under $\Phi$. This essentially means that the following $\mathcal{X}$-homeomorphisms

$$\Phi_0 : \partial\mathcal{A}_0 \to \partial\mathcal{B}_0$$

$$\Phi_1 : \partial\mathcal{A}_1 \to \partial\mathcal{B}_1$$

are such that $\alpha_A\big|_{\partial\mathcal{A}_0} = \alpha_B\Phi_0$ and $\alpha_A\big|_{\partial\mathcal{A}_1} = \alpha_B\Phi_1$.

Finally, we define 2-morphisms in $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$ as homotopies up to an isotopy on the boundary. Let us spell it out. Consider two 1-morphisms $\mathcal{A}, \mathcal{B} : \mathcal{K} \to \mathcal{M}$. A 2-morphism $\Phi : \mathcal{A} \to \mathcal{B}$ is a triple $(\phi, \alpha, \gamma)$ where $\phi : A \times [0,1] \to X$ and $\alpha : \partial A \times [0,1] \to (-K) \sqcup M$ are continuous maps such that :

(i) $(A, \phi_0, \alpha_0) = \mathcal{A}$,

(ii) $(A, \phi_t, \alpha_t)$ is a 1-morphism from $\mathcal{K}$ to $\mathcal{M}$ for each $t \in [0, 1]$.

(iii) $\gamma : (A, \phi_1, \alpha_1) \to \mathcal{B}$ is an $\mathcal{X}$-homeomorphism of 1-morphisms.

The composition of 2-morphisms is defined by cutting the interval $[0, 1]$ in half. This composition is associative only up to homotopy (on $[0, 1]$). This is where 3-morphisms appear! Similarly the trivial homotopy is identity 2-morphism only up to homotopy and each 2-morphism admits an inverse up to homotopy. One can fix this by choosing homotopy classes of 2-morphisms but we do not do it because our interest in 2-morphisms is temporary.

We say that two 1-morphisms $\mathcal{A}, \mathcal{B} : \mathcal{K} \to \mathcal{M}$ are *equivalent* if there exists a 2-morphism from $\mathcal{A}$ to $\mathcal{B}$ and in that case we write $\mathcal{A} \sim \mathcal{B}$. We say 0-morphisms $\mathcal{M} = (M, f_M, p_M)$ and $\mathcal{K} = (K, f_K, p_K)$ are *isomorphic* if there are 1-morphisms $\mathcal{A} = (A, f_A, \alpha_A)$ and $\mathcal{B} = (B, f_B, \alpha_B)$ from $\mathcal{M}$ to $\mathcal{K}$ and $\mathcal{K}$ to $\mathcal{M}$ respectively, such that $I_{\mathcal{K}} \sim \mathcal{A} \circ \mathcal{B}$ and $I_{\mathcal{M}} \sim \mathcal{B} \circ \mathcal{A}$. In this case we say $\mathcal{B}$ is an inverse of $\mathcal{A}$.

Using an $\mathcal{X}$-homeomorphisms $\Psi$ between $\mathcal{M} = (M, f_M, p_M)$ and $\mathcal{K} = (K, f_K, p_K)$ we construct a 1-morphism $\mathcal{A}_\Psi = (M \times I, f_A, \alpha_A^\Psi)$ from $\mathcal{M}$ to $\mathcal{K}$ where $f_A$ is the projection of the characteristic map of $M$ and the boundary map $\alpha_A^\Psi$ is simply identity on in-boundary and $\Psi$ on out-boundary. We will generalise this concept and call them *cylinders* in the next subsection.

**Proposition 3.3.1** *If $\Psi$ and $\Phi$ are isotopic $\mathcal{X}$-homeomorphisms between $\mathcal{M} = (M, f_M, p_M)$ and $\mathcal{K} = (K, f_K, p_K)$ then the 1-morphisms $\mathcal{A}_\Psi$ and $\mathcal{A}_\Phi$ are equivalent in $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$.*

PROOF: Since $\Psi$, $\Phi : \mathcal{M} \to \mathcal{K}$ are isotopic, there exists a continuous path say $H$ in the space of $\mathcal{X}$-homeomorphisms from $\mathcal{M}$ and $\mathcal{K}$ connecting $\Psi$ and $\Phi$. Now for $\mathcal{A}_\Psi$ and $\mathcal{A}_\Phi$ to be equivalent we need to establish a 2-morphism $(\phi, \alpha, \gamma)$ between them. If the space of these 1-morphisms is $A = M \times I$, the space of the two morphism is $A \times I$. As $M \times I$ is the space for the two 1-morphisms , result is trivial i.e. the $A \times I$ and the isotopy on the out-boundary. Let us spell it out. The projection of the characteristic map of $M$ gives

$\phi : A \times I \to X$. The boundary map $\alpha : \partial A \times I \to -M \sqcup K$ is given as

$$\alpha_0 : \partial A_0 \times I \to -M \sqcup K$$

defined by $\alpha_0 = id_M$, for all $t \in I$ and,

$$\alpha_1 : \partial A_1 \times I \to -M \sqcup K$$

defined by $\alpha_1 = H_t$, for all $t \in I$. Further, $\gamma : (A, \phi_1, \alpha_1) \to (A, f_M, \alpha_A^\Phi)$ is an identity homeomorphism. □

This proposition filters the action of the full homeomorphism group and only the mapping class group survives. The next proposition shows that, up to isomorphism, $\mathcal{M} = (M, f_M, p_M)$ depends only on the homotopy class of $f_M$. More precisely,

**Proposition 3.3.2** *Let $\mathcal{M}_f = (M, f, p_M)$ and $\mathcal{M}_g = (M, g, p_M)$ be 0-morphisms in $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$ such that there is a homotopy between $f$ and $g$, then $(M, f, p_M)$ is isomorphic to $(M, g, p_M)$ in $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$.*

Proof: Let $F : M \times I \to X$ be a homotopy between $f$ and $g$. Then we have a 1-morphism $\mathcal{A} = (M \times I, F, \alpha_A)$ between $\mathcal{M}_f$ and $\mathcal{M}_g$ which is identity on boundary. Defining $\overline{F} : M \times I \to X$ as

$$\overline{F}(m, t) = F(m, 1 - t)$$

we have another 1-morphism $\mathcal{B} = (M \times I, \overline{F}, \alpha_B)$ between $\mathcal{M}_g$ and $\mathcal{M}_f$, which is also identity on boundary. We need to show $\mathcal{A}$ and $\mathcal{B}$ are inverse to each other. We need to compose $\mathcal{A}$ and $\mathcal{B}$ and show the composition to be identity. Define $H : (M \times [0,1]) \times I \to X$ as follows: 
$$H\big((m,t),s\big) = \begin{cases} F\Big(m, \dfrac{2t}{1-s}\Big) & ;\ 0 \le t \le \dfrac{1-s}{2} \\ g(m) & ;\ \dfrac{1-s}{2} \le t \le \dfrac{1+s}{2} \\ \overline{F}\Big(m, \dfrac{2t+2}{1-s}\Big) & ;\ \dfrac{1+s}{2} \le t \le 1. \end{cases}$$

Note that,
$$H_0(t,x) = \begin{cases} F(m, 2t) & ;\ 0 \le t \le \dfrac{1}{2} \\ \overline{F}(m, 2t+2) & ;\ \dfrac{1}{2} \le t \le 1. \end{cases}$$

which gives $H_0(m,t) = \overline{F} \sqcup F(m,t)$. Further, $H_1(m,t) = g(m)$, $\forall\ t \in$ [0,1] which is

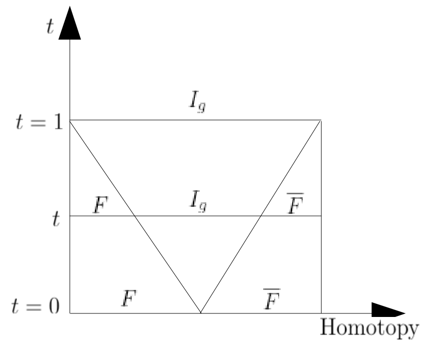



Figure 3.2: The homotopy between $F \sqcup \overline{F}$ and $I_g$.

the identity homotopy $I_g$ at $g$. Thus $H$ so defined is an isotopy between $\overline{F} \sqcup F$ and $I_g$. Then Proposition 3.3.1, gives,

$$(M \times [0,1], \overline{F} \sqcup F, id_M) = (M \times [0,1], I_g, id_M).$$

Thus we have finally

$$\begin{aligned}
\mathcal{B} \circ \mathcal{A} &= (M \times [0,1], \overline{F}, id_M) \circ (M \times [0,1], F, id_M) \\
&= (M \times [0,1], \overline{F} \sqcup F, id_M) \\
&= (M \times I, I_g, id_M) \\
&= I_{\mathcal{M}}.
\end{aligned}$$

Similar procedure as above with suitable changes will give the equality in other direction. □

One can actually work with $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$ but it is very big to work with and not even a category. We would like to take a chance to reduce it even further and construct a category $\mathcal{X} - \widehat{\mathrm{Cob}}_n$. Its objects are 0-morphisms of $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$. Its morphisms are equivalence classes of 1-morphisms in $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$. We call a 1-morphism $\mathcal{A} = (A, f_A, \alpha_A)$ a *cylinder* if $A \cong M \times I$ as topological manifolds, refer Section 3.4.4.

**Proposition 3.3.3** *$\mathcal{X} - \widehat{\mathrm{Cob}}_n$ is a category.*

Proof: Composition of morphisms has been defined above. Consider 1-morphisms, $\mathcal{A} : \mathcal{K} \to \mathcal{L}$, $\mathcal{B} : \mathcal{L} \to \mathcal{M}$ and $\mathcal{C} : \mathcal{M} \to \mathcal{N}$. We know there is an orientation preserving homeomorphisms of manifolds $(A \sqcup B) \sqcup C \to A \sqcup (B \sqcup C)$, and thus it is clear $(\mathcal{A} \circ \mathcal{B}) \circ \mathcal{C}$ and $\mathcal{A} \circ (\mathcal{B} \circ \mathcal{C})$ are equivalent in $\mathcal{X} - \widetilde{\mathrm{Cob}}_n$. We know that a cylinder is homotopic to its base. The Figure 3.3 below shows the collapsing of the cylinder to its base giving the identity morphism. Thus a 1-morphism $\mathcal{A} : \mathcal{M} \to \mathcal{M}$ such that $A = M \times I$ with identity

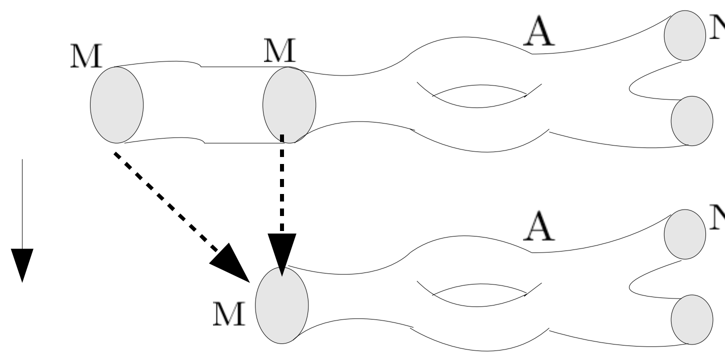



Figure 3.3: The identity axiom in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$.

on the boundary gives the identity morphism in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$.

□

Taking disjoint union of 0-morphisms (defined earlier) provides a monoidal structure to $\mathcal{X} - \widehat{\mathrm{Cob}}_n$. For any $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$, the three natural isomorphisms

$$\widehat{a}_{\mathcal{L},\mathcal{M},\mathcal{N}} : (\mathcal{L} \sqcup \mathcal{M}) \sqcup \mathcal{N} \to \mathcal{L} \sqcup (\mathcal{M} \sqcup \mathcal{N})$$

$$\widehat{\lambda} : (I \sqcup \mathcal{M}) \to \mathcal{M}$$

$$\widehat{\rho} : (\mathcal{M} \sqcup I) \to \mathcal{M}$$

clearly satisfy the coherence conditions expressing the fact that the tensor operation is associative and has left and right identity. Moreover, the commutativity constraint

$$\widehat{c}_{\mathcal{M},\mathcal{N}} : \mathcal{M} \sqcup \mathcal{N} \to \mathcal{N} \sqcup \mathcal{M}$$

forms a natural family of isomorphisms in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ that satisfy the hexagonal axioms. This equips $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ with a symmetric tensor product. We have the following result.

**Theorem 3.3.4** *$\mathcal{X} - \widehat{\mathrm{Cob}}_n$ is a rigid symmetric monoidal category.*

Proof: The tensor product or monoidal product has been discussed above. It is associate and has left and right identity. Rigidity in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ is as follows: for a 0-morphism $\mathcal{M} = (M, f, p)$ we have already defined $\mathcal{M}^* = (-M, f, p)$. The rigidity structure $\eta_{\mathcal{M}}$ is the 1-morphism given by $(I \times M, 1_f, id_M)$ viewed as a cobordism: $\emptyset \to \mathcal{M} \sqcup \mathcal{M}^*$, and the rigidity structure $\epsilon_{\mathcal{M}}$ is given by 1-morphism $(I \times M, 1_f, id_M)$ viewed as a cobordism: $\mathcal{M} \sqcup \mathcal{M}^* \to \emptyset$.

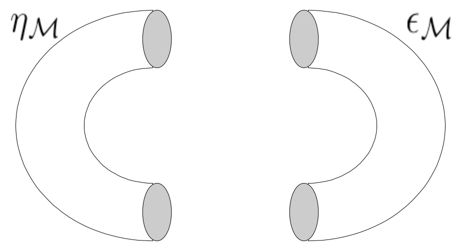



Figure 3.4: The duality structure in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$.

The maps $\eta_{\mathcal{M}}$ and $\epsilon_{\mathcal{M}}$ are defined by cylinders as shown below in Figure 3.4. For an $\mathcal{X}$-homeomorphism $\phi : \mathcal{K} \to \mathcal{L}$ we have $\phi^* = \phi_-^{-1}$ where $\phi_-$ is the homeomorphism $-K \to -L$ induced by $\phi$. For a 1-morphism $\mathcal{A} : \mathcal{K} \to \mathcal{L}$, its dual $\mathcal{A}^*$ has the same underlying base space as of $\mathcal{A}$ but with opposite orientation and now viewed as a cobordism $\mathcal{L}^* \to \mathcal{K}^*$. Thus if the boundary map of $\mathcal{A}$ is $\alpha_{\mathcal{A}} : \partial\mathcal{A} \to (-K) \sqcup L$, then the boundary map $\alpha_{\mathcal{A}^*} : \partial\mathcal{A}_- \to (-L) \sqcup K$ for $\mathcal{A}^*$ is the same as $\alpha_{\mathcal{A}}$ but with oppositely oriented domain and codomain. □

**Examples.** For **n=0**, $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ is a category with countable collection of points as objects and maps sending these points to the base point of $X$. Homotopy classes of loops in $X$ form the morphisms in this category.

For **n=1**, countable union of loops in $X$ are the objects. The morphisms are cobordisms with maps into $X$ (3-dimensional manifolds without boundary or circles as bound-

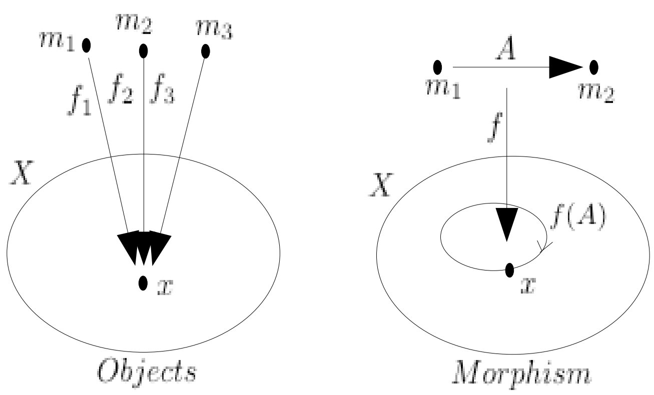



Figure 3.5: Objects and morphisms between them for n=0.

aries! ).

Let us now state the most important theorem of this section which essentially redefines an HQFT as a monoidal functor.

**Theorem 3.3.5** *(i) If $(Z,\tau)$ is an $(n+1)$-dimensional $\mathcal{X}$-HQFT taking values in a monoidal category $\mathcal{C}$, then $\widetilde{Z} : \mathcal{M} \mapsto Z(M)$ and $\mathcal{A} \mapsto Z(A)$ defines a monoidal functor from $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ to $\mathcal{C}$.*

*(ii) If $(Z,\tau)$ is an $(n+1)$-dimensional symmetric $\mathcal{X}$-HQFT with values in a symmetric monoidal category $\mathcal{C}$, then $\widetilde{Z} : \mathcal{M} \mapsto Z(M)$ and $\mathcal{A} \mapsto Z(A)$ defines a symmetric monoidal functor from $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ to $\mathcal{C}$.*

*(iii) If $W : \mathcal{X} - \widehat{\mathrm{Cob}}_n \longrightarrow \mathcal{C}$ is a monoidal functor then the assignment $\mathcal{M} \mapsto W(\mathcal{M})$ and $\mathcal{A} \mapsto W(\mathcal{A})$, where $\mathcal{M}$ and $\mathcal{A}$ respectively are 0 and 1-morphisms, defines an $(n+1)$-dimensional $\mathcal{X}$-HQFT $(\widehat{W},\tau)$ with values in $\mathcal{C}$.*

*(iv) Suppose $\mathcal{C}$ is a symmetric monoidal category. If $W : \mathcal{X} - \widehat{\mathrm{Cob}}_n \longrightarrow \mathcal{C}$ is a symmetric monoidal functor then the assignment $\mathcal{M} \mapsto W(\mathcal{M})$ and $\mathcal{A} \mapsto W(\mathcal{A})$, where $\mathcal{M}$ and $\mathcal{A}$ respectively are 0 and 1-morphisms, defines an $(n+1)$-dimensional symmetric $\mathcal{X}$-HQFT $(\widehat{W},\tau)$ with values in $\mathcal{C}$.*

PROOF:

(i) Let $(Z, \tau)$ be an $(n+1)$-dimensional $\mathcal{X}$-HQFT taking values in a monoidal category $\mathcal{C}$. Let us define a functor $\widetilde{Z} : \mathcal{X} - \widehat{\mathrm{Cob}_n} \to \mathcal{C}$ as $\widetilde{Z} : \mathcal{M} \mapsto Z(\mathcal{M})$ and $\widetilde{Z} : \mathcal{A} \mapsto Z(\mathcal{A})$. The Axioms (ii) and (iii) of an HQFT gives the coherence morphisms $\phi_{\mathcal{M},\mathcal{N}} : \widetilde{Z}(M) \otimes \widetilde{Z}(\mathcal{N}) \to \widetilde{Z}(\mathcal{M} \otimes \mathcal{N})$ and $\widetilde{Z}(\emptyset) \to I_{\mathcal{C}}$. The axioms for the coherence maps would follow automatically owing to the natural $\mathcal{X}$-homemorphisms between the $n$-manifolds.

(ii) We have already shown in part (i) that $\widetilde{Z}$ is a monoidal functor. Axiom (ix) of a symmetric HQFT makes $\widetilde{Z}$ a braided monoidal functor. Since $\mathcal{X} - \mathrm{Cob}_n$ and $\mathcal{C}$ are symmetric monoidal categories, $\widetilde{Z}$ is a symmetric monoidal functor.

(iii) Consider a monoidal functor $W : \mathcal{X} - \widehat{\mathrm{Cob}_n} \to \mathcal{C}$. We want to define an $(n+1)$-dimensional $\mathcal{X}$-HQFT $(\widehat{W}, \tau)$ with values in $\mathcal{C}$. Let us define it explicitly:

$$\widehat{W}(\mathcal{M}) = W(\mathcal{M});\ \mathcal{M} \text{ is a 0-morphism,}$$

$$\tau(\mathcal{A}) : W(\mathcal{M}) \xrightarrow{W(\mathcal{A})} W(\mathcal{N});\ \mathcal{A} : \mathcal{M} \to \mathcal{N} \text{ is a 1-morphism.}$$

Given an $\mathcal{X}$-homeomorphism $f : \mathcal{M} \to \mathcal{N}$ we have a 1-morphism $\mathcal{A}_{\mathcal{M}} = (M \times I, f_A, \alpha_{\mathcal{A}})$, with $f_A = f_m \times 1$, $\alpha^0_{\mathcal{A}} = id_M$ and $\alpha^1_{\mathcal{A}} = f : M \to N$. The inverse of $\mathcal{A}_{\mathcal{M}}$ is again a 1-morphism given by $\mathcal{B}_{\mathcal{M}} = (M \times I, f_B, \alpha_{\mathcal{B}})$ with $f_B = f_m \times 1$, $\alpha^0_{\mathcal{B}} = f : M \to N$ and $\alpha^1_{\mathcal{B}} = id_M$. Then $\mathcal{B}_{\mathcal{M}} \circ \mathcal{A}_{\mathcal{M}} = id_{\mathcal{M}}$ and similarly we have the identity on $\mathcal{N}$. Thus, the morphism in $\mathcal{C}$ corresponding to $\mathcal{A}_{\mathcal{M}}$ is an isomorphism. So, we set

$$f_{\#} = W(\mathcal{A}_{\mathcal{M}}) : \widehat{W}(\mathcal{M}) \xrightarrow{W(\mathcal{A}_{\mathcal{M}})} \widehat{W}(\mathcal{N});\ f : \mathcal{M} \to \mathcal{N} \text{ is an } \mathcal{X}\text{-homeomorphism.}$$

We now discuss the axioms of an $\mathcal{X}$-HQFT. For any $\mathcal{X}$-homeomorphisms $f : \mathcal{L} \to \mathcal{M}$ and $f' : \mathcal{M} \to \mathcal{N}$, $(f'f)_{\#} = f'_{\#} f_{\#}$ since $\widehat{W}$ is a functor. Further, $\widehat{W}$ being a monoidal functor, we have $\widehat{W}(\mathcal{M} \sqcup \mathcal{N}) \cong \widehat{W}(\mathcal{M}) \otimes \widehat{W}(\mathcal{N})$ and $\widehat{W}(\emptyset) \cong I_{\mathcal{C}}$. Axiom (iv) follows from the way we have set up the category $\mathcal{X} - \widehat{\mathrm{Cob}_n}$ with morphisms as equivalence classes of 1-morphisms up to 2-morphisms in $\mathcal{X} - \widetilde{\mathrm{Cob}_n}$. Axiom (v) is obvious. As discussed before, we can consider any $\mathcal{X}$-homeomorphism $f : \mathcal{M} \to \mathcal{N}$

as a 1-morphism $(M \times I, f_M, \alpha)$, where $\alpha^0 = id_M$ and $\alpha^1 = f : M \to N$. Then the gluing axiom becomes the composition of morphisms which is preserved by a functor. The identity morphism $\mathcal{A}_M$ for any 0-morphism $\mathcal{M}$ gives the Axiom (vii). Finally the last axiom is a part of the definition of the category $\mathcal{X} - \widehat{\mathrm{Cob}}_n$.

(iv) $(\widehat{W}, \tau)$ is an $\mathcal{X}$-HQFT by part (iii). Axiom (ix) of an HQFT follows from the fact that $W$ is a symmetric monoidal functor.

□

So far we have worked with a single space $\mathcal{X}$ fixed in the background. Suppose now we have two spaces $\mathcal{X}$, $\mathcal{Y}$ which interact with each other via a continuous map. Let us explore the interaction between the corresponding symmetric monoidal categories $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ and $\mathcal{Y} - \widehat{\mathrm{Cob}}_n$; and the $\mathcal{X}$-HQFTs and $\mathcal{Y}$-HQFTs that one can construct from $\mathcal{X}$ and $\mathcal{Y}$. Suppose $\Psi : \mathcal{X} \to \mathcal{Y}$ is a continuous map of pointed path-connected topological spaces, $\mathcal{X} = (X, x)$ and $\mathcal{Y} = (Y, y)$. If $\mathcal{M} = (M, f_M, p_M)$ is a 0-morphism in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$, then $\mathcal{M}_{\mathcal{Y}} = (M, f_M^{\mathcal{Y}} = \Psi f_M, p_M)$ will be a 0-morphism in $\mathcal{Y} - \widehat{\mathrm{Cob}}_n$. In the same way, if $\mathcal{A} = (A, f_A, \alpha_A) : \mathcal{M} \to \mathcal{K}$ is a 1-morphism in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ then $\mathcal{A}_{\mathcal{Y}} = (A, f_A^{\mathcal{Y}} = \Psi f_A, \alpha_A) : (M, f_M^{\mathcal{Y}}, p_M) \to (K, f_K^{\mathcal{Y}}, p_K)$ is a 1-morphism in $\mathcal{Y} - \widehat{\mathrm{Cob}}_n$. These assignments induces a functor

$$\widehat{\Psi}_* : \mathcal{X} - \widehat{\mathrm{Cob}}_n \to \mathcal{Y} - \widehat{\mathrm{Cob}}_n$$

such that $\mathcal{M} \mapsto \mathcal{M}_{\mathcal{Y}}$ and $\mathcal{A} \mapsto \mathcal{A}_{\mathcal{Y}}$. Note that

$$\begin{aligned}
\mathcal{M}_{\mathcal{Y}} \sqcup \mathcal{N}_{\mathcal{Y}} &= \big(\mathcal{M} \sqcup \mathcal{N}, f_M^{\mathcal{Y}} \sqcup f_N^{\mathcal{Y}}, p_M \sqcup p_N\big) \\
&= \big(\mathcal{M} \sqcup \mathcal{N}, \Psi f_M \sqcup \Psi f_N, p_M \sqcup p_N\big) \\
&= \big(\mathcal{M} \sqcup \mathcal{N}, \Psi(f_M \sqcup f_N), p_M \sqcup p_N\big) \\
&= \big(\mathcal{M} \sqcup \mathcal{N}, \Psi(f_{M \sqcup N}), p_M \sqcup p_N\big) \\
&= \big(\mathcal{M} \sqcup \mathcal{N}, f_{M \sqcup N}^{\mathcal{Y}}, p_M \sqcup p_N\big) \\
&= (\mathcal{M} \sqcup \mathcal{N})_{\mathcal{Y}}.
\end{aligned}$$

Similarly, $\Big((\mathcal{L} \sqcup \mathcal{M}) \sqcup \mathcal{N}\Big)_{\mathcal{Y}} = (\mathcal{L}_{\mathcal{Y}} \sqcup \mathcal{M}_{\mathcal{Y}}) \sqcup \mathcal{N}_{\mathcal{Y}}$. This makes $\widehat{\Psi}_*$ a strict monoidal functor. Further, observe that the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{\mathcal{Y}} \sqcup \mathcal{N}_{\mathcal{Y}} \arrow[r] \arrow[d, "="'] & \mathcal{N}_{\mathcal{Y}} \sqcup \mathcal{M}_{\mathcal{Y}} \arrow[d, "="] \\
(\mathcal{M} \sqcup \mathcal{N})_{\mathcal{Y}} \arrow[r] & (\mathcal{N} \sqcup \mathcal{M})_{\mathcal{Y}}
\end{tikzcd}$$

relates the braidings in the two categories. Indeed the functor is symmetric in nature (hexagonal axioms are obvious). Thus $\widehat{\Psi}_*$ is a strict symmetric monoidal functor.

Now let $\Psi$ and $\Phi$ be continuous maps $\mathcal{X} \to \mathcal{Y}$ and $H$ a homotopy between them. If $(M, f_M, p_M)$ is a 0-morphism over $\mathcal{X}$, then $(M, \Psi f_M, p_M)$ and $(M, \Phi f_M, p_M)$ are 0-morphisms over $\mathcal{Y}$. On the other hand, the map $H \circ (f_M \times I_{[0,1]}) : M \times [0,1] \to Y$ defined by

$$(m, t) \mapsto H(f_M(m), t)$$

is a homotopy between $\Psi f_M$ and $\Phi f_M$, and by proposition (3.3.1), there is an isomorphism between $(M, \Psi f_M, p_M)$ and $(M, \Phi f_M, p_M)$. This induces a natural isomorphism

$$\eta : \widehat{\Psi}_* \to \widehat{\Phi}_*.$$

**Definition 3.3.6** *Given $n \geq 0$, an $n$-homotopy equivalence between topological spaces is a continuous map which induces isomorphisms on homotopy groups $\pi_k(X, x), k \leq n$.*

The existence of an $n$-homotopy equivalence from $X$ to $Y$ is a reflexive and transitive relation, but not symmetric. Thus we state that $X$ and $Y$ are weakly $n$-homotopy equivalent if there exists a zigzag of $n$-homotopy equivalences $X \leftarrow\ \rightarrow \leftarrow \cdots \rightarrow Y$. Homotopy $n$-types are the equivalence classes of a weak $n$-homotopy equivalence relation. Thus, if two spaces have the *same homotopy $n$-type* then their homotopy groups agree up to $n$.

Let us consider the category $\mathcal{Z}_{n+1}(\mathcal{X}, \mathcal{C})$ of $(n+1)$-dimensional $\mathcal{X}$-HQFTs taking values in a monoidal category $\mathcal{C}$. The objects are $(n+1)$-dimensional $\mathcal{X}$-HQFTs with values in $\mathcal{C}$. A morphism $(Z, \tau) \to (Z', \tau')$ in this category is a family of morphisms $\{\rho_{\mathcal{M}} : Z_{\mathcal{M}} \to Z'_{\mathcal{M}}\}$ where $\mathcal{M}$ runs over 0-morphisms in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ such that $\rho_{\emptyset} = I_{\mathcal{C}}$ ; for disjoint 0-morphisms $\mathcal{M}$ and $\mathcal{N}$, we have $\rho_{\mathcal{M} \sqcup \mathcal{N}} = \rho_{\mathcal{M}} \otimes \rho_{\mathcal{N}}$; the natural square diagrams associated with $\mathcal{X}$-homeomorphisms of 0-morphisms and with 1-morphisms are commutative. Note that if the monoidal category $\mathcal{C}$ in the background is clear and there is no ambiguity, then we may simply use $\mathcal{Z}_{n+1}(\mathcal{X})$.

Observe that HQFTs can also be pushed forward along the maps between the target spaces. Given a map $\Psi : \mathcal{X} \to \mathcal{Y}$ of path-connected pointed spaces we can transform any $\mathcal{X}$-HQFT into a $\mathcal{Y}$-HQFT both taking values in the same domain monoidal category $\mathcal{C}$. It suffices to compose the characteristic maps with $\Psi$. This induces a functor $\Psi_* : \mathcal{Z}_{n+1}(\mathcal{X}, \mathcal{C}) \to \mathcal{Z}_{n+1}(\mathcal{Y}, \mathcal{C})$. We finally state the main result of the section. Note that the credit of this theorem goes to Turaev who has established a similar theorem for any $(n+1)$-type path connected topological spaces giving equivalence of the category of HQFTs over them. For the proof see Theorem 2.2.1, [Tur99].

**Theorem 3.3.7** *Let $n \geq 0$. If two path-connected pointed spaces $\mathcal{X}$ and $\mathcal{Y}$ have the same homotopy $(n+1)$-type, then there exists a zigzag of functors $\mathcal{Z}_{n+1}(\mathcal{X}, \mathcal{C}) \leftarrow\ \rightarrow\ \leftarrow \cdots \rightarrow \mathcal{Z}_{n+1}(\mathcal{Y}, \mathcal{C})$ which are equivalences of categories; thus*

$$\mathcal{Z}_{n+1}(\mathcal{X}, \mathcal{C}) \sim \mathcal{Z}_{n+1}(\mathcal{Y}, \mathcal{C})$$

*are equivalent categories.*

Proof: Since $\mathcal{X}$ and $\mathcal{Y}$ have the same homotopy $(n+1)$-type, they are connected by a zigzag of $n$-homotopy equivalences $\mathcal{X} \leftarrow\ \rightarrow\ \leftarrow \cdots \rightarrow \mathcal{Y}$ which induces isomorphisms on all homotopy groups up to $n+1$. □

## 3.4 Crossed Systems

Inspired by the work done by Turaev on HQFTs [Tur99], we define Turaev $G$-crossed system for a group $G$. We then define cylinders in an appropriate category and finally we give one of the main results of this chapter that circles and cylinders form a Turaev Crossed system. Let us first define this appropriate category (will call it $\mathcal{X} - \mathrm{Cob}_n$) as follows.

### 3.4.1 Reduced category : $\mathcal{X} - \mathrm{Cob}_n$

Let us choose a collection $\{X_\alpha\}$ of representatives of isomorphism classes of connected 0-morphisms in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$. Let $\mathcal{X} - \mathrm{Cob}_n = < X_\alpha >$ be the smallest full subcategory containing these objects such that it is closed under disjoint unions. Then $\mathcal{X} - \mathrm{Cob}_n$ is a fully faithful monoidal subcategory of $\mathcal{X} - \widehat{\mathrm{Cob}}_n$. Note that for any object $\mathcal{M}$ in $\mathcal{X} - \mathrm{Cob}_n$, and for all $i \geq 1$, the objects corresponding to the disjoint union of $\mathcal{M}$ taken $i$ times, can be related to an associahedron of $(i-2)$-dimensional polyhedron whose $j$-dimensional cells, for $0 \leq j \leq i-2$, are $(i-j-2)$ pairs of brackets between $i$ copies of $\mathcal{M}$.

- For $i = 1$, it is simply $\mathcal{M}$ itself. Consider it as the empty set.
- For $i = 2$, it is the tensor product of $\mathcal{M}$ with itself. It can be interpreted as a single point

$$\mathcal{M} \sqcup \mathcal{M}.$$

- For $i = 3$, we will have two tensor cubes. It is the usual associativity in $\mathcal{X} - \mathrm{Cob}_n$

$$(\mathcal{M} \sqcup \mathcal{M}) \sqcup \mathcal{M} \xrightarrow{a} \mathcal{M} \sqcup (\mathcal{M} \sqcup \mathcal{M}).$$

  It can be interpreted as a single interval.

- The fourth associahedron is the pentagon which expresses the different ways a disjoint union of four $\mathcal{M}$'s may be bracketed

$$\begin{tikzcd}
& & (\mathcal{M} \sqcup \mathcal{M}) \sqcup (\mathcal{M} \sqcup \mathcal{M}) \arrow[drr, "a"] & & \\
\Big((\mathcal{M} \sqcup \mathcal{M}) \sqcup \mathcal{M}\Big) \sqcup \mathcal{M} \arrow[urr, "a"] \arrow[dr, "a \otimes 1"'] & & & & (\mathcal{M} \sqcup \Big(\mathcal{M} \sqcup (\mathcal{M} \sqcup \mathcal{M})\Big) \\
& \Big(\mathcal{M} \sqcup (\mathcal{M} \sqcup \mathcal{M})\Big) \sqcup \mathcal{M} \arrow[rr, "a"] & & \mathcal{M} \sqcup \Big((\mathcal{M} \sqcup \mathcal{M}) \sqcup \mathcal{M}\Big) \arrow[ur, "1 \otimes a"'] &
\end{tikzcd}$$

  and so on.

Thus we have an inclusion functor

$$\mathcal{G} : \mathcal{X} - \mathrm{Cob}_n \hookrightarrow \mathcal{X} - \widehat{\mathrm{Cob}}_n$$

which is such that each object $X$ of $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ is isomorphic to an object of $\mathcal{X} - \mathrm{Cob}_n$. We wish to formulate the inverse of this functor so that the inclusion functor $\mathcal{G}$ becomes an equivalence of categories and $\mathcal{X} - \mathrm{Cob}_n$ becomes equivalent to $\mathcal{X} - \widehat{\mathrm{Cob}}_n$. Let us explain how to construct the inverse functor and hence the equivalence.

**Theorem 3.4.1** *The functor* $\mathcal{G} : \mathcal{X} - \mathrm{Cob}_n \hookrightarrow \mathcal{X} - \widehat{\mathrm{Cob}}_n$ *is a symmetric monoidal equivalence.*

PROOF: For each object $X$ in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$, let us choose an object $\mathcal{F}X$ and an isomorphism $\theta_X : X \to \mathcal{F}X$ of $\mathcal{X} - \mathrm{Cob}_n$ which sends $x \mapsto \mathcal{F}x$ and its inverse sends $y \mapsto \hat{y}$. Suppose $\mathcal{A}$ is a morphism between $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$. Then $\mathcal{F}(\mathcal{A})$ is given by the composition of morphisms

$$\mathcal{F}\mathcal{M} \xrightarrow{\theta_{\mathcal{M}}^{-1}} \mathcal{M} \xrightarrow{\mathcal{A}} \mathcal{N} \xrightarrow{\theta_{\mathcal{N}}} \mathcal{F}\mathcal{N}.$$

Thus we construct an inverse functor

$$\mathcal{F} : \mathcal{X} - \widehat{\mathrm{Cob}}_n \to \mathcal{X} - \mathrm{Cob}_n$$

so that $\theta$ becomes a natural isomorphism $\theta : 1 \cong \mathcal{G}\mathcal{F}$. Moreover, $\mathcal{F}\mathcal{G} \cong 1$, and so $\mathcal{G}$ is an equivalence. Indeed, $\mathcal{G}$ is a strict monoidal equivalence with the coherence maps

$$\mathcal{G}\mathcal{M} \otimes \mathcal{G}\mathcal{N} \longrightarrow \mathcal{G}(\mathcal{M} \otimes \mathcal{N})$$

$$I \longrightarrow \mathcal{G}I$$

being simply identities. Let us explore whether the inverse functor $\mathcal{F}$ is also a strict monoidal. The answer is affirmative but not completely. We show that $\mathcal{F}$ is a monoidal functor, which may not be strict in general. Let us set up the coherence maps as the composition

$$\Phi_{\mathcal{M},\mathcal{N}} : \mathcal{F}\mathcal{M} \otimes \mathcal{F}\mathcal{N} \xrightarrow{\theta_{\mathcal{M}}^{-1} \otimes \theta_{\mathcal{N}}^{-1}} \mathcal{M} \otimes \mathcal{N} \xrightarrow{\theta_{\mathcal{M}\otimes\mathcal{N}}} \mathcal{F}(\mathcal{M} \otimes \mathcal{N})$$

$$\Phi : \emptyset \to \mathcal{F}\emptyset.$$

of isomorphisms in $\mathcal{X}-\mathrm{Cob}_n$. Since there is only one empty set, $\Phi$ is simply an identity and thus the coherence axioms for $\emptyset$ becomes trivial. For the equivalence $\mathcal{F}$ to be monoidal, we first proceed to show the diagram (2.4) commutes. We need to show

$$a\Big[(\theta_{\mathcal{L}} \sqcup \theta_{\mathcal{M}}) \sqcup \theta_{\mathcal{N}}\Big] = \Big[\theta_{\mathcal{L}} \sqcup (\theta_{\mathcal{M}} \sqcup \theta_{\mathcal{N}})\Big]\widehat{a} \tag{3.7}$$

where $a$ and $\widehat{a}$ are the assocciativities of $\mathcal{X} - \mathrm{Cob}_n$ and $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ respectively. For any $l, m, n$ respectively in $L, M, N$, $x = (l, 0, 0) \sqcup (m, 0, 1) \sqcup (n, 1) \in (L \sqcup M) \sqcup N$. Then LHS of equation (3.7) when hit with $x$ becomes

$$\begin{aligned} a\Big[\big(\theta_{\mathcal{L}}(l,0,0) \sqcup \theta_{\mathcal{M}}(m,0,1)\big) \sqcup \theta_{\mathcal{N}}(n,1)\Big] &= a\Big[(\mathcal{F}l,0,0) \sqcup (\mathcal{F}m,0,1) \sqcup (\mathcal{F}n,1)\Big] \\ &= (\mathcal{F}l,0) \sqcup (\mathcal{F}m,1,0) \sqcup (\mathcal{F}n,1,1). \end{aligned}$$

On the other hand, RHS of equation (3.7) after being hit by $x$ becomes

$$\begin{aligned} \Big[\theta_{\mathcal{L}} \sqcup (\theta_{\mathcal{M}} \sqcup \theta_{\mathcal{N}})\Big]\widehat{a}\Big[(l,0,0) \sqcup (m,0,1) \sqcup (n,1)\Big] &= \Big[\theta_{\mathcal{L}} \sqcup (\theta_{\mathcal{M}} \sqcup \theta_{\mathcal{N}})\Big]\Big[(l,0) \sqcup (m,1,0) \sqcup (n,1,1)\Big] \\ &= \theta_{\mathcal{L}}(l,0) \sqcup \theta_{\mathcal{M}}(m,1,0) \sqcup \theta_{\mathcal{N}}(n,1,1) \\ &= (\mathcal{F}l,0) \sqcup (\mathcal{F}m,1,0) \sqcup (\mathcal{F}n,1,1). \end{aligned}$$

Next, we proceed to show the diagrams in (2.5) commutes. Its suffice to show that the equation

$$\theta_{\mathcal{M}}\widehat{\rho}(\theta_{\mathcal{M}}^{-1} \sqcup 1) = \rho$$

holds true where $\rho$ and $\widehat{\rho}$ are the left units of $\mathcal{X} - \mathrm{Cob}_n$ and $\mathcal{X} - \widehat{\mathrm{Cob}}_n$ respectively. For any $m$ in the base space of $\mathcal{F}\mathcal{M}$, $x = (m, 0)$ is an element in the base space of $\mathcal{F}\mathcal{M} \sqcup \emptyset$.

Then we have

$$\begin{aligned}\theta_{\mathcal{M}}\widehat{\rho}(\theta_{\mathcal{M}}^{-1} \sqcup 1)(m,0) &= \theta_{\mathcal{M}}\widehat{\rho}\big(\theta_{\mathcal{M}}^{-1}(m),0\big)\\ &= \theta_{\mathcal{M}}\theta_{\mathcal{M}}^{-1}(m)\\ &= m\end{aligned}$$

and $\rho(m,0) = m$. Thus by Proposition (2.1.2), $\mathcal{F}$ is a monoidal functor. Finally we want to show the diagram (2.6) commutes. That is,

$$(\theta_{\mathcal{N}}^{-1} \sqcup \theta_{\mathcal{M}}^{-1})c = \widehat{c}(\theta_{\mathcal{M}}^{-1} \sqcup \theta_{\mathcal{N}}^{-1}).$$

For any $(x,0) \sqcup (y,1) \in \mathcal{FM} \sqcup \mathcal{FN}$,

$$\begin{aligned}(\theta_{\mathcal{N}}^{-1} \sqcup \theta_{\mathcal{M}}^{-1})c\big((x,0) \sqcup (y,1)\big) &= (\theta_{\mathcal{N}}^{-1} \sqcup \theta_{\mathcal{M}}^{-1})\big((y,0) \sqcup (x,1)\big)\\ &= \theta_{\mathcal{N}}^{-1}(y,0) \sqcup \theta_{\mathcal{M}}^{-1}(x,1)\\ &= (\widehat{y},0) \sqcup (\widehat{x},1)\end{aligned}$$

and,

$$\begin{aligned}\widehat{c}(\theta_{\mathcal{M}}^{-1} \sqcup \theta_{\mathcal{N}}^{-1})\big((x,0) \sqcup (y,1)\big) &= \widehat{c}\Big(\theta_{\mathcal{M}}^{-1}(x,0) \sqcup \theta_{\mathcal{N}}^{-1}(y,1)\Big)\\ &= \widehat{c}\Big((\widehat{x},0) \sqcup (\widehat{y},1)\Big)\\ &= (\widehat{y},0) \sqcup (\widehat{x},1).\end{aligned}$$

Thus by Corollary (2.1.3), $\mathcal{F}$ is a symmetric monoidal functor

$$\mathcal{X} - \widehat{\mathrm{Cob}}_n \longrightarrow \mathcal{X} - \mathrm{Cob}_n.$$

□

Let us now define a Turaev crossed system in any symmetric monoidal category.

### 3.4.2 Turaev crossed system

Let $G$ be any group. Consider a Frobenius $G$-graded system $\mathcal{F} = (A, \mu, \eta, \Delta, \epsilon)$ in a symmetric monoidal category $\mathcal{C}$. Then $A = \big(A_g, \mu_{g,h}, \eta\big)$ is a $G$-algebra. Let $\phi = \{\phi_g | g \in G\}$, such that for each $g \in G$,

$$\phi_g = \epsilon\mu_{g,g^{-1}} : A_g \otimes A_{g^{-1}} \xrightarrow{\mu_{g,g^{-1}}} A_1 \xrightarrow{\epsilon} I$$

is the non-degenerate form associated with the system. Let $\varphi = \{\varphi_{g,h} : A_g \to A_{hgh^{-1}} | g, h \in G\}$ be a set of morphisms in $\mathcal{C}$.

**Definition 3.4.2** *Given a symmetric monoidal category $\mathcal{C}$, and a group $G$, we define a Turaev $G$-crossed system in $\mathcal{C}$ (or a Turaev crossed system over $G$) as an ordered quintuple $\mathcal{T} = (A, \mu, \eta, \phi, \varphi)$ where $(A, \mu, \eta)$ has a Frobenius $G$-graded structure and $\mu$, $\varphi$ and $\phi$ satisfy the following set of axioms which are encapsulated in the commutative diagrams drawn below:*

**(3.1)** $\varphi_{f,gh} = \varphi_{hfh^{-1},g} \circ \varphi_{f,h}$ ;

$$\begin{tikzcd}
A_f \arrow[r, "\varphi_{f,h}"] \arrow[dr, "\varphi_{f,gh}"'] & A_{hfh^{-1}} \arrow[d, "\varphi_{hfh^{-1},g}"] \\
 & A_{ghfh^{-1}g^{-1}}
\end{tikzcd}$$

**(3.2)** $\varphi_{fg,h} \circ \mu_{f,g} = \mu_{hfh^{-1},hgh^{-1}}\big(\varphi_{f,h} \otimes \varphi_{g,h}\big)$ ;

$$\begin{tikzcd}
A_f \otimes A_g \arrow[r, "\mu_{f,g}"] \arrow[d, "\varphi_{f,h}\otimes\varphi_{g,h}"'] & A_{fg} \arrow[d, "\varphi_{fg,h}"] \\
A_{hfh^{-1}} \otimes A_{hgh^{-1}} \arrow[r, "\mu_{hfh^{-1},hgh^{-1}}"'] & A_{h(fg)h^{-1}}
\end{tikzcd}$$

**(3.3)** *$\varphi$ preserves the pairing $\phi$, that is, $\phi_{hfh^{-1}}(\varphi_{f,h} \otimes \varphi_{f^{-1},h}) = \phi_f$. The following* *diagram exhibits this property*

$$\begin{tikzcd} A_f \otimes A_{f^{-1}} \arrow[r, "\varphi_{f,h}\otimes\varphi_{f^{-1},h}"] \arrow[dr, "\phi_f"'] & A_{hfh^{-1}} \otimes A_{hf^{-1}h^{-1}} \arrow[d, "\phi_{hfh^{-1}}"] \\ & I \end{tikzcd}$$

**(3.4)** *$\varphi_{g,g} = id$, for all $g \in G$;*

**(3.5)** *The following diagrams commute.*

$$\begin{tikzcd} A_f \otimes A_g \arrow[r, "\varphi_{f,g}\otimes 1"] \arrow[d, "\tau"'] & A_{gfg^{-1}} \otimes A_g \arrow[d, "\mu_{gfg^{-1},g}"] \\ A_g \otimes A_f \arrow[r, "\mu_{g,f}"'] & A_{gf} \end{tikzcd} \qquad \begin{tikzcd} A_f \otimes A_g \arrow[r, "1\otimes\varphi_{g,f^{-1}}"] \arrow[d, "\tau"'] & A_f \otimes A_{f^{-1}gf} \arrow[d, "\mu_{f,f^{-1}gf}"] \\ A_g \otimes A_f \arrow[r, "\mu_{g,f}"'] & A_{gf} \end{tikzcd}$$

**(3.6)** *For $f, g \in G$, let $h = fgf^{-1}g^{-1}$. Consider the following composition of maps*

$$b : A_h \otimes A_f \xrightarrow{1\otimes\varphi_g} A_h \otimes A_{gfg^{-1}} \xrightarrow{\mu} A_f$$

$$c : A_h \otimes A_g \xrightarrow{\mu} A_{hg} \xrightarrow{\varphi_{f^{-1}}} A_g$$

*The trace of the above two compositions, which is a map from $A_h$ to $I$, is equal. (Refer Section 2.1).*

**Lemma 3.4.3** *Let $\mathcal{T}$ be a Turaev $G$-crossed system. Then,*

**(a)** *$\varphi_{*,1}$ is identity.*

**(b)** *$\varphi_{g,g^{-1}}$ is identity for all $g \in G$.*

**(c)** *$\phi_g$ is symmetric for all $g \in G$, i.e. $\phi_g = \phi_{g^{-1}}\tau$.*

Proof: Substituting $f = g$ and $h = 1$ in axiom (3.1), we get $\varphi_{g,g} = \varphi_{g,g} \circ \varphi_{g,1}$. Then using axiom (3.4) which says $\varphi_{g,g} = id$, part (a) holds. Part (b) follows from axiom (3.4) together with axiom (3.1). For every $g \in G$, symmetricity of $\phi_g$ is implied by substituting $f = g^{-1}$ in axiom (3.5) and then using $\varphi_{g,g^{-1}} = id$. □

Note that in case $G = 1$, Turaev crossed system is simply a Frobenius system (without grading).

We define a category, $\mathcal{T}(G) = \mathcal{T}(G;\mathcal{C})$, whose objects are Turaev crossed $G$-systems in a symmetric monoidal category $\mathcal{C}$. A morphism $\mathcal{T} \to \mathcal{T}'$ in this category is a collection of morphisms in $\mathcal{C}$ mapping each $A_f$ to $A'_f$, preserving the unit and the non-degenerate form associated with the system and commuting with the multiplication and the action of $G$. We will need this category only later in the Section 3.5.

### 3.4.3 K(G,n+1) case

The idea of this subsection and the next section is to give a brief description of the basic structure of the cobordism category when the target space is a pointed $K(G, n+1)$ space.

Take $\mathcal{X}$ to be the Eilenberg-Maclane space $K(G, n+1)$ for some abelian group $G$ and $n \geq 1$. In this case the objects in $\mathcal{X} - \mathrm{Cob}_n$ are disjoint unions of the chosen representatives, say $\mathcal{M}$, where $M$ is a compact manifold without a boundary of dimension $n$ and we have

$$[M, X] \cong H^{n+1}(M; G) \cong 0. \tag{3.8}$$

Here, $[M, X]$ is the group of all homotopies from $M$ to $X$. Using the Proposition 3.3.2 we can assume that the characteristic maps of our objects are trivial, that is, they send everything to the base point $x$ of $X$. For morphisms $\mathcal{A}$ in $\mathcal{X} - \mathrm{Cob}_n$ we have

$$[\mathcal{A}, \mathcal{X}]_{\partial A} \cong [(A, \partial A), \mathcal{X}] \cong H^{n+1}((A, \partial A); G). \tag{3.9}$$

Also we know by Poincare duality,

$$H^{n+1}((A, \partial A); G) \cong H_0(A; G). \tag{3.10}$$

and then by universal coefficient theorem this amounts to $H_0(A) \otimes G$. This is nothing but the connected components of $A$ labelled with elements of $G$. Thus in this case ($n \geq 1$), $\mathcal{X} - \mathrm{Cob}_n$ is essentially the same category $\mathrm{Cob}_n$ with labelled components.

### 3.4.4 K(G,1) case

Let us set the following notation. An $(n, \mathcal{X})$-manifold is an $\mathcal{X}$-manifold of dimension $n$ which is same as a 0-morphism in $\mathcal{X} - \widehat{\mathrm{Cob}}_n$. In particular, for $n = 1$, we shall call a 0-morphism an $\mathcal{X}$-circle. Now let us look into $\mathcal{X} - \mathrm{Cob}_1$. Any object $\mathcal{M} = (M, f_M, p_M)$ in this category is isomorphic to $\sqcup_r (M^r_\epsilon, g_r \in G)_r$, where each $M^r_\epsilon$ is a (1,$\mathcal{X}$)-manifold, $\epsilon \in \{\pm 1\}$ indicates the orientation of $M^r_\epsilon$ and the element $g_r$ of $G$, represented by the restriction of $f_M$ to $M^r_\epsilon$, is the homotopy class of maps from $M^r_\epsilon$ into X.

On the other hand, every (1,$\mathcal{X}$)-manifold is homeomorphic to a disjoint union of copies of the $\mathcal{X}$-circles where the $\mathcal{X}$-circles : $(S^1_+; g)$, $(S^1_-; g)$ are simply a circle $S^1$ with the standard positive and negative orientations and the homotopy class of the map into $\mathcal{X}$ is given by $g \in G$. This means that the objects of $\mathcal{X} - \mathrm{Cob}_1$ are made up of disjoint unions of $(S^1_+; g^{-1})$ and $(S^1_-; h)$. Note that $(S^1_-; g)$ is the dual of $(S^1_+; g)$. Also $(S^1_-; g)$ is isomorphic to $(S^1_+; g)$. Thus $(S^1_+; g^{-1})$ is the dual of $(S^1_+; g)$, which may not be isomorphic in $\mathcal{X} - \mathrm{Cob}_1$.

Before we go deeper in to $\mathcal{X} - \mathrm{Cob}_1$ let us set the following notation. If $\mathcal{A} = (A, f_A, \alpha_A)$ is a morphism in $\mathcal{X} - \mathrm{Cob}_n$, we shall refer $\partial A_0$ as in-boundary and $\partial A_1$ as out-boundary of $A$ considered as a cobordism such that $\partial A = \partial A_0 \sqcup \partial A_1$. Further, we write $M_+$ for a component $M \subseteq \partial A$ with orientation induced from $A$ and $M_-$ for $M$ with opposite orientation. Let $(M^r_-, g_r \in G)_r$ be components of $\partial A$ whose orientations are opposite to the one induced from $A$. Here $g_r$ is represented by the restriction of $f_A$ to $M^r_-$. Let $(M^s_+, h_s \in G)$ be the components of $\partial A$ whose orientations are induced from $A$. Here $h_s$ is represented by the restriction of $f_A$ to $M^s_+$. Thus we can view $A$ as a cobordism between $\partial A_0 = \sqcup_r (M^r_-, g_r)$ and $\partial A_1 = \sqcup_s (M^s_+, h_s)$.

We know that any two dimensional compact oriented $X$-manifold, or an $X$-cobordism can be constructed using one of the following three basic structures : a 2-disc; a 2-disc with one hole (annulus), or a 2-disc with two holes. Let us denote $D_\epsilon$ a 2-disc viewed as a cobordism between empty set and $S^1_+$. Thus,

$$\partial D_\epsilon = \epsilon S^1_\epsilon.$$

For $\epsilon = \pm$, $D_\epsilon$ is then a cobordism with oriented pointed boundary between $\emptyset$ and $S^1_\epsilon$. By definition, the homotopy class of the map $f_D : D_\epsilon \to X$ is determined by the homotopy class $g \in G$ represented by the loop $f|_{S^1_\epsilon}$. See the figure (3.6) below:

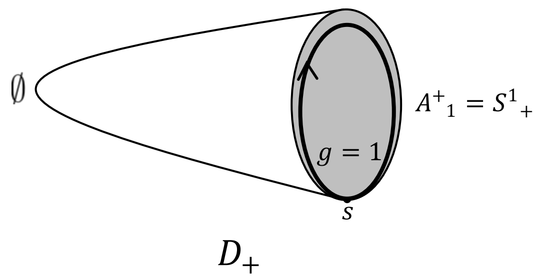



Figure 3.6: An $\mathcal{X}$-disc $D_+$

The loop $f|_{S^1_\epsilon}$ represents $g$. Denote by $D_\epsilon(g)$ the $\mathcal{X}$-disc $D_\epsilon$ endowed with the map to $X$ corresponding to the homotopy class $g \in G$ and identity on the boundary.

Note that there is only one homotopy class of maps $D_+ \to X$. This implies $D_+$ is an $\mathcal{X}$-morphism from $\emptyset$ to $(S^1_+, 1)$ and consequently $D_-$ is an $\mathcal{X}$-morphism from $(S^1_-, 1)$ to $\emptyset$. Thus $D_-$ is the dual morphism of $D_+$.

Let $C$ denote the annulus $S^1 \times [0,1]$. We fix an orientation of $C$. Set $C^0 = S^1 \times 0 \subset \partial C$ and $C^1 = S^1 \times 1 \subset \partial C$. Let us provide $C^0, C^1$ with base points $c^0 = s \times 0, c^1 = s \times 1$, respectively, where $s \in S^1$. For any signs $\epsilon, \mu = \pm$, $C_{\epsilon,\mu}$ is a cobordism from $C^0_\epsilon$ to $C^1_\mu$. This is an annulus with oriented pointed boundary. By definition,

$$\partial C_{\epsilon,\mu} = (\epsilon C^0_\epsilon) \cup (\mu C^1_\mu).$$

Now the homotopy classes of the map $f_C : C_{\epsilon,\mu} \to X$ is determined by the homotopy classes $g, h \in G$ represented by the loops $f|_{C^0_\epsilon}$ and $f|_{s\times[0,1]}$, respectively. See the picture below:

Here the interval $[0,1]$ is oriented from 0 to 1. The loop $f|_{C^1_\mu}$ represents $(h^{-1}g^{-\epsilon}h)^\mu$. Let us call $C_{\epsilon,\mu}(g,h)$ the $\mathcal{X}$-annulus $C_{\epsilon,\mu}$ which is a cylinder from $C^0_\epsilon$ to $C^1_\mu$ endowed with the map to $X$ corresponding to the pair $g, h \in G$ and identity on the boundary. For calculations, we shall be writing $C^0_\epsilon(g)$ and $C^1_\mu\big((h^{-1}g^{-\epsilon}h)^\mu\big)$ to indicate the boundary components along with the loops representing them.

Let $P$ be an oriented 2-disc with two holes, also called a pant. Denote the boundary components of $P$ by $L, M, N$ and provide them with base points $l, m, n$, respectively. For

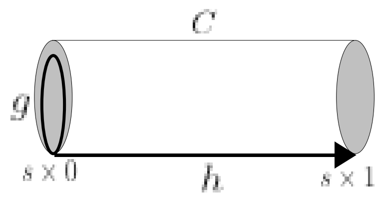



Figure 3.7: An $\mathcal{X}$-annulus $C_{-+}(g, h)$

any signs $\epsilon, \mu, \nu = \pm$ we denote by $P_{\epsilon,\mu,\nu}$ the quadruple $(P, L_\epsilon, M_\mu, N_\nu)$. This is a 2-disc with two holes and oriented pointed boundary. By definition,

$$\partial P_{\epsilon,\mu,\nu} = (\epsilon L_\epsilon) \cup (\mu M_\mu) \cup (\nu N_\nu).$$

To analyse the homotopy classes of maps $P_{\epsilon,\mu,\nu} \to X$, we fix two proper embedded arcs $nl$ and $nm$ in $P$ starting from $n$ to $l, m$ and mutually disjoint except of course in the endpoint $n$. See the picture below :

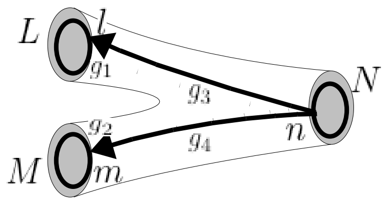



Figure 3.8: An $\mathcal{X}$-pant $P_{--+}(g_1, g_2, g_3, g_4)$.

To every map $f : P_{\epsilon,\mu,\nu} \to X$ we assign the homotopy classes of the loops $f|_{L_\epsilon}, f|_{M_\mu}, f|_{nl}, f|_{nm}$. This establishes a bijective correspondence between the set of homotopy classes of maps $D_{\epsilon,\mu,\nu} \to X$ and $G^4$. For any $g_1, g_2, g_3, g_4 \in G$, let us call $P_{\epsilon,\mu,\nu}(g_1, g_2, g_3, g_4)$ the $\mathcal{X}$-pant $P_{\epsilon,\mu,\nu}$ endowed with the map to $X$ corresponding to $g_1, g_2, g_3, g_4$. Note that the loops $f|_{L\epsilon}, f|_{M\mu}, f|_{N\nu}$ represent the classes $g_1$, $g_2$, $(g_4 g_1^{-\epsilon} g_4^{-1} g_3 g_2^{-\mu} g_3^{-1})^\nu$, respectively. For calculations, we shall refer $L$, $M$, $N$ related to any pant $P_{--+}(g_1, g_2, g_3, g_4)$ as $(L_\epsilon, g_1)$,

$(M_\epsilon, g_2)$ and $\big(N_\epsilon, (g_4 g_1^{-\epsilon} g_4^{-1} g_3 g_2^{-\mu} g_3^{-1})^\nu\big)$ respectively to indicate the boundary components of the pant along with the loops representing them.

Let us make use of the notation to state the following crucial points about $\mathcal{X} - \mathrm{Cob}_1$:

(i) The $\mathcal{X}$-circle $(S^1_\epsilon, g)$ is simply a circle $S^1_\epsilon$ with the standard positive or negative orientation given by $\epsilon \in \{\pm 1\}$ and the homotopy class of the map into $\mathcal{X}$ is given by $g \in G$.

(ii) There are two possible $\mathcal{X}$-discs: $D_+$, $D_-$ with $D_-$ as the dual of $D_+$ in $\mathcal{X} - \mathrm{Cob}_1$. $\mathcal{X}$-cobordism : $\emptyset \to (S^1_+, 1)$ with its dual morphism given as $D_- : (S^1_-, 1) \to \emptyset$.

(iii) There are four possible $\mathcal{X}$-annuli $C_{\epsilon,\mu}(g_1, g_2)$ for $\epsilon, \mu \in \{\pm\}$ which are cylinders from $(S^1_\epsilon; g_1)$ to $(S^1_{\epsilon,\mu}; (g_2^{-1} g_1^{-\epsilon} g_2)^\mu)$. Thus we have :

- $C_{-+}(g, 1) : (S^1_-, g) \to (S^1_+, g)/C_{+-}(g, 1) : (S^1_+, g) \to (S^1_-, g)$.
- $C_{--}(g, 1) : (S^1_-, g) \sqcup (S^1_-, g^{-1}) \to \emptyset$.
- $C_{++}(g^{-1}, 1) : \emptyset \to (S^1_+, g^{-1}) \sqcup (S^1_+, g)$.

(iv) There are eight possible $\mathcal{X}$-pants for different values of $\epsilon, \mu, \nu \in \{\pm\}$ which are cobordisms with boundaries in $\{L = \big(S^1_\epsilon, g_1\big), M = \big(S^1_\mu, g_2\big), N = \big(S^1_\nu, (g_3 g_1^{-\epsilon} g_3^{-1} g_4 g_2^{-\mu} g_4^{-1})^\nu\big)\}$.

Some basic ones as:

- $P_{--+}(g, h, 1, 1) : (S^1_-; g) \sqcup (S^1_-; h) \to (S^1_+; gh)$.
- $P_{---}(g, h, 1, 1) : (S^1_-; g) \sqcup (S^1_-; h) \sqcup (S^1_-; k) \to \emptyset$, where $k = (gh)^{-1}$

These morphisms mentioned above are the basic morphisms that generate the whole of the category $\mathcal{X} - \mathrm{Cob}_1$.

Let us denote $(S^1_-; g)$ as $A_g$, and the collection $(A_g)_{g \in G}$ as $A$. We call the collection $A$ as *circles* in $\mathcal{X} - \mathrm{Cob}_1$. Note that since $(S^1_+, g) \cong (-S^1_-, g)$ in $\mathcal{X} - \mathrm{Cob}_1$, thus $C_{-+}(g, 1)$ is essentially identity morphism of $A_g$. We denote it as $1_g : A_g \to A_g$. For each $g \in G$, let us set $A^*_g = (S^1_-; g^{-1})$ and let $A^* = \{(A_{g^{-1}})^* : g \in G\}$. Then for each $g \in G$ $A_g$ is isomorphic to $(A_{g^{-1}})^*$. Thus $A$ is isomorphic to $A^*$ in $\mathcal{X} - \mathrm{Cob}_1$. Now the $\mathcal{X}$-pant $P_{--+}(f, g, 1, 1)$ is an $\mathcal{X}$-morphism in $\mathcal{X} - \mathrm{Cob}_1$ between $L \cup M$ and $N$ where $L = (S^1_-, f)$, $M = (S^1_-, g)$

and $N = (S^1_+, fg)$. We set and denote this morphism in $\mathcal{X} - \mathrm{Cob}_1$ as

$$\mu_{f,g} : A_f \otimes A_g \longrightarrow A_{fg} \tag{3.11}$$

for $f, g \in G$. We claim that $\{\mu_{f,g}\}_{f,g\in G}$ provides an associative multiplication for the collection $A$. The multiplication in $A$ also provides it with an $A$-module structure.

The $\mathcal{X}$-disc $D_+$ gives the unit. Let us denote it as

$$\eta : I \longrightarrow A_1, \text{ ; and} \tag{3.12}$$

the $\mathcal{X}$-disc $D_-$ gives the counit. Let us denote it as

$$\epsilon : A_1 \longrightarrow I. \tag{3.13}$$

For any $f \in G$, the $\mathcal{X}$-cylinder $C_{--}(f, 1)$ is an $\mathcal{X}$-morphism between $(S^1_-, f) \sqcup (S^1_-, f^{-1})$ and $\emptyset$, given by the composition of $\mathcal{X}$-pant $P_{--+}(f, f^{-1}, 1, 1)$ with $\mathcal{X}$-disc $D_-$. Let us denote the collection of these morphisms over $G$ by $\phi = \{\phi_f\}$ which gives pairings on components of $A$ as:

$$\phi_f : A_f \otimes A_{f^{-1}} \longrightarrow I \tag{3.14}$$

given by $\phi_f = \epsilon\mu_{f,f^{-1}}$.

Similarly, the copairing can be defined using the cylinder $C_{++}(f, 1)$ which is an $\mathcal{X}$-morphism between $\emptyset$ and $(S^1_+, f) \sqcup (S^1_+, f^{-1})$ given by the composition of $\mathcal{X}$-disc $D_+$, with $\mathcal{X}$-pant $P_{-++}(1, f, 1, 1)$. Let us denote the collection of copairings on $A$ as $\widetilde{\phi} = \{\widetilde{\phi}_f\}$ given by:

$$\widetilde{\phi}_f : I \longrightarrow A_f \otimes A_{f^{-1}}.$$

The figure below exhibits the pairing and its copairing.

Note that as shown in Theorem 3.2.8, the multiplication and the form will define a comultiplication, say, $\Delta$. Thus with this collection of morphisms, $\{A_g\}$ forms a Frobenius $G$-graded system in $\mathcal{X} - \mathrm{Cob}_1$. We have the following result:

**Proposition 3.4.4** *Circles in $\mathcal{X} - \mathrm{Cob}_1$ form a Frobenius $G$-graded system with $\mathcal{X} =$*

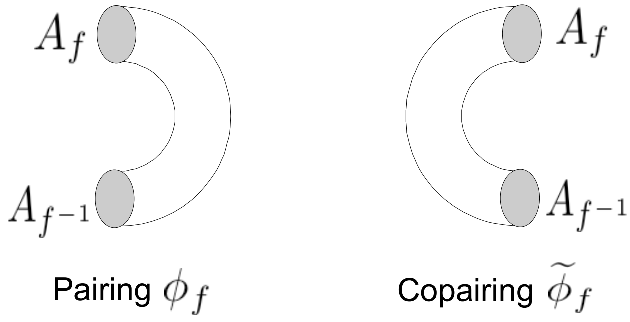



Figure 3.9: The non-degenerate symmetric form on circles.

$\big(K(G,1),x\big)$*; multiplication* $\mu = \{\mu_{f,g}\}$*; unit* $\eta$ *and pairing* $\phi = \{\phi_g\}$ *described above.*

Proof: The collection $A = (A_g)_{g\in G}$ of circles in $\mathcal{X} - \mathrm{Cob}_1$ as discussed above has a set of morphisms attached to it: $\mu_{f,g} : A_f \otimes A_g \longrightarrow A_{fg}$ and $\eta : I \to A_1$, for $f, g, 1 \in G$. We argue the associativity of the multiplication as follows. The gluing of $P_{--+}(f,g,1,1)$ to $P_{--+}(fg,h,1,1)$ along $L : (S^1_+, fg) \cong (-S^1_-, fg) : N$, gives the same $\mathcal{X}$-morphism as the gluing of $P_{--+}(g,h,1,1)$ to $P_{--+}(f,gh,1,1)$ along an $\mathcal{X}$-homeomorphism $M : (S^1_+, gh) \cong (-S^1_-, gh) : N$.

The unit is as follows. There is only one homotopy class of maps $D_+ \to X$. We denote this $\mathcal{X}$-morphism in $\mathcal{X} - \mathrm{Cob}_1$ as $\eta : I \to A_1$. This element is a right unit for $A$ because the gluing of $D_+$ to $P_{--+}(f,1,1,1)$ along an $\mathcal{X}$-homeomorphism $\partial D_+ \cong M_-$ yields $C_{-+}(f,1)$. Similarly it will be a left unit when we glue $D_+$ to $P_{--+}(1,f,1,1)$ along an $\mathcal{X}$-homeomorphism $\partial D_+ \cong L_-$.

Non-degeneracy of the pairing (form): Gluing the cylinder $C_{--}(g,1)$ to $C_{++}(g^{-1},1)$ along $(S^1_-, g^{-1}) \cong (-S^1_+, g^{-1})$, we obtain the $\mathcal{X}$-morphism $C_{-+}(g,1)$ which we know gives identity morphism of $A_g$. Figure (3.10) exhibits this topologically.

Thus using equivalence under gluing of cylinders in $\mathcal{X} - \mathrm{Cob}_1$, we get:

$$\big(\widetilde{\phi}_{g^{-1}} \otimes 1_g\big)\big(1_g \otimes \phi_g\big) = 1_g$$

and similarly we get:

$$\big(1_{g^{-1}} \otimes \widetilde{\phi}_g\big)\big(\phi_{g^{-1}} \otimes 1_{g^{-1}}\big) = 1_{g^{-1}},$$

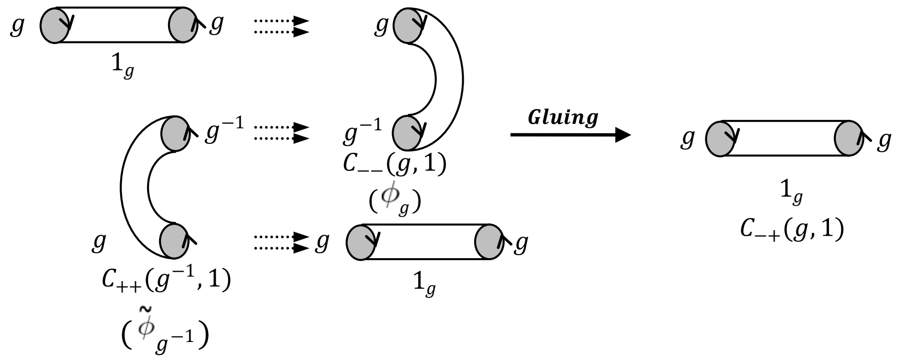



Figure 3.10: Non-degeneracy of the form

which are essentially the non-degeneracy conditions for the pairing and its copairing $\phi_g$, $\widetilde{\phi}_g$.

Assume that $fgh = 1$. Gluing $C_{--}(fg,1)$ to $P_{--+}(f,g,1,1)$ along the $\mathcal{X}$-homeomorphism $N_+ = (S^1_+, fg) \cong (-S^1_-, fg) = (C^1_-, fg)$, we obtain the $\mathcal{X}$-morphism $P_{---}(f,g,1,1)$ which gives

$$\phi_{fg} : A_f \otimes A_{gh} \longrightarrow I.$$

Now $P_{--+}(f,g,1,1)$ and $P_{--+}(g,f,1,1)$ are $\mathcal{X}$-homeomorphic in $\mathcal{X} - \widetilde{\mathrm{Cob}}_1$. The homeomorphism maps the boundary components say, $L, M, N$ of the first $\mathcal{X}$-cylinder onto the boundary $N, L, M$ of the second $\mathcal{X}$-cylinder respectively. Thus they are in the same equivalence class of 1-morphism in $\mathcal{X} - \widetilde{\mathrm{Cob}}_1$. Hence, we have $\phi_{fg} = \phi_{gf}$. This implies the form is symmetric. Thus using Theorem 3.2.8, the proof is completed. □

### 3.4.5 Cylinders and $\mathcal{X}$-Cylinders

Let $\mathcal{A} : \mathcal{M} \to \mathcal{N}$ be a morphism in $\mathcal{X} - \mathrm{Cob}_n$. We call $\mathcal{A}$ a *cylinder* if $A \cong M \times I$ as topological manifolds. And $\mathcal{A}$ is an $\mathcal{X}$-*cylinder* of $M$ or simply an $\mathcal{X}$-*cylinder* if $\mathcal{A}$ is $\mathcal{X}$-homeomorphic to $(M \times I, f_M, \alpha)$, with $\alpha$ given as identity on one-end and boundary map of $A$ at the other end. Clearly, an $\mathcal{X}$-cylinder is a cylinder, but a cylinder is not necessarily an $\mathcal{X}$-cylinder. For example, consider the handle $\emptyset \to S^1 \sqcup S^1$. The concept of cylinders and $\mathcal{X}$-cylinders have two applications, or rather two directions to go about. One is considering the mapping class group of $\mathcal{M}$ containing all the equivalent classes of the $\mathcal{X}$-cylinders of $M$. And the other direction is to enrich the collection of circles in $\mathcal{X}$ with a structure of Turaev crossed system using $\mathcal{X}$-cylinders of circles. Let us denote the set of representatives of equivalence classes of 1-morphisms between $\mathcal{M}$ and $\mathcal{K}$ as $\mathrm{Hom}\,(\mathcal{M},\mathcal{K})$, then $\mathcal{A}_\Psi \in \mathrm{Hom}\,(\mathcal{M},\mathcal{K})$ and proposition (3.3.1) implies that $\mathcal{A}_\Psi$ and $\mathcal{A}_\Phi$ represent the same element in $\mathrm{Hom}\,(\mathcal{M},\mathcal{K})$ when $\Psi$ and $\Phi$ are isotopic. Note that $\mathcal{A}_\psi \in \mathrm{Hom}\,(\mathcal{M},\mathcal{M})$, when $\psi$ is a self $\mathcal{X}$-homeomorphism of $\mathcal{M}$.

Let $\mathrm{Homeo}^{\mathcal{X}}(\mathcal{M})$ be the group of self $\mathcal{X}$-homeomorphisms of $\mathcal{M}$ with compact open topology. Let $\mathrm{Homeo}_0^{\mathcal{X}}(\mathcal{M})$ be the subset of $\mathrm{Homeo}^{\mathcal{X}}(\mathcal{M})$ consisting of all $\mathcal{X}$-homeomorphisms isotopic to the identity morphism on $\mathcal{M}$. It is easy to verify that $\mathrm{Homeo}_0^{\mathcal{X}}(\mathcal{M})$ is in fact a normal open subgroup in $\mathrm{Homeo}^{\mathcal{X}}(\mathcal{M})$. The factor group

$$\mathrm{MCG}^{\mathcal{X}}(\mathcal{M}) = \mathrm{Homeo}^{\mathcal{X}}(\mathcal{M})/\mathrm{Homeo}_0^{\mathcal{X}}(\mathcal{M})$$

is the mapping class group of $\mathcal{M}$. The elements of this group are the isotopy classes of self $\mathcal{X}$-homeomorphisms of $\mathcal{M}$. Note that $\mathcal{X}$-cylinders define a semigroup homomorphism from $\mathrm{MCG}^{\mathcal{X}}(\mathcal{M})$ to $\mathrm{Hom}\,(\mathcal{M},\mathcal{M})$ given by $[\psi] \mapsto \mathcal{A}_\psi$ and the composition is given by : $[\psi \circ \phi] \mapsto \mathcal{A}_{\psi\circ\phi} = \mathcal{A}_\psi \circ \mathcal{A}_\phi$.

Let us now work in the other direction to provide the circles with a structure of a Turaev $G$-crossed system.

Using these morphisms, we conclude the section with the following result:

**Theorem 3.4.5** *Let $G$ be a group and $\mathcal{X}$ a $K(G;1)$ space. Then in $\mathcal{X} - \mathrm{Cob}_1$, cylinders define a Turaev $G$-crossed system on circles.*

Proof: We have discussed in the Proposition 3.4.4 that circles form a Frobenius $G$-graded system in $\mathcal{X} - \mathrm{Cob}_1$. The cylinder $C_{--}(g,1)$ defines a form $\phi_g : A_g \otimes A_{g^{-1}} \to I$ on circles. The information of $G$-action on circles is carried by the cylinder $C_{-+(g,h^{-1})}$, i.e.

$$\phi_g : A_g \otimes A_{g^{-1}} \xrightarrow{C_{--}(g,1)} I$$

$$\varphi_{g,h} : A_g \xrightarrow{C_{-+}(g,h^{-1})} A_{hgh^{-1}}$$

We show the axioms of a Turaev crossed system in the following seven easy steps:

(i) $\varphi_{f,gh} = \varphi_{hfh^{-1},g} \circ \varphi_{f,h}$ is depicted in Figure 3.11. Observe that the gluing of $C_{-+}(g,h^{-1})$ to $C_{-+}(hgh^{-1},k^{-1})$ yields $C_{-+}(g,(kh)^{-1})$. Then the axiom holds true

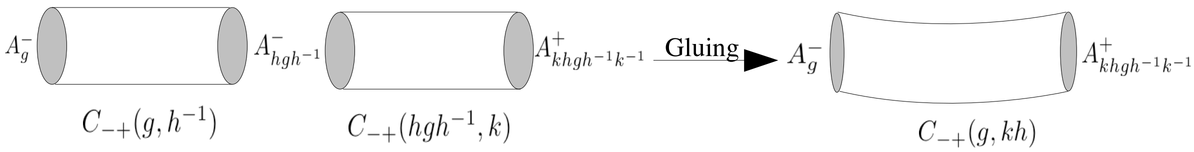



Figure 3.11: The non-degenerate symmetric form on circles.

using the composition of the cylinders in $\mathcal{X} - \mathrm{Cob}_1$.

(ii) The way we have defined the action, we have

$$\varphi_{g,1} : A_g \longrightarrow A_g$$

is an identity morphism on $A_g$. This is so because $A_g = (S^1_-, g)$ and the $\mathcal{X}$-morphism $\varphi_{g,1}$ is the $\mathcal{X}$-cylinder $C_{-+}(g, 1)$ mapping $S^1$ to $S^1$ as an identity map.

(iii) We show that the action is an algebra morphism. Gluing $C_{-+}(fg, h^{-1})$ to $P_{--+}(f, g, 1, 1)$ along $(S^1_-, fg) = (C^0_+, fg)$, we obtain the $\mathcal{X}$-morphism $P_{--+}(f, g, h, h)$.

Similarly , gluing $C_{-+}(f, h^{-1}) \cup C_{-+}(g, h^{-1})$ to $P_{--+}(hfh^{-1}f, hgh^{-1}, 1, 1)$ along the circles $(S^1_-, hfh^{-1}) = (L_-, hfh^{-1})$ and $(S^1_-, hgh^{-1}) = (M_-, hgh^{-1})$ respectively, we again obtain the same $\mathcal{X}$-morphism $P_{--+}(f, g, h, h)$. Therefore,

$$\varphi_{fg,h} \circ \mu_{f,g} = \mu_{hfh^{-1},hgh^{-1}}\big(\varphi_{f,h} \otimes \varphi_{g,h}\big).$$

(iv) We argue that the action preserves form as follows. Assume that $g = f^{-1}$. Now, gluing $C_{-+}(f, h^{-1}) \cup C_{-+}(g, h^{-1})$ to the cylinder $C_{--}(hfh^{-1}, 1)$ along the circles

$$(S^1_-, hfh^{-1}) = (L_-, hfh^{-1}) \text{ and,}$$

$$(S^1_-, hgh^{-1}) = (M_-, hgh^{-1})$$

respectively, we obtain the $\mathcal{X}$-morphism $C_{--}(f, 1)$. Thus,

$$\phi_{hfh^{-1}}(\varphi_{f,h} \otimes \varphi_{g,h}) = \phi_f.$$

(v) The Dehn twist along the circle $S^1 \times (1/2) \subset C_{-+}(f,1)$ yields an $\mathcal{X}$-homeomorphism between $C_{-+}(f,1)$ and $C_{-+}(f,f)$. Thus they are in the same equivalence class of isomorphism in $\mathcal{X} - \mathrm{Cob}_1$. Hence

$$\varphi_{f,f} = \varphi_{f,1} = \mathrm{id} : A_f \longrightarrow A_f.$$

(vi) Consider a self $\mathcal{X}$-homeomorphism $\zeta$ of the $(2,\mathcal{X})$-pants $P$ which is the identity on $N$ and which permutes $(L,l)$ and $(M,m)$. We choose $\zeta$ so that $\zeta(nm) = nl$ and $\zeta(nl)$ is an embedded arc leading from $n$ to $m$ and homotopic to the product of four arcs: $nl$, $\partial L$, $(nl)^{-1}$, $nm$. An easy computation shows that $\zeta$ is an $\mathcal{X}$-homeomorphism from $P_{--+}(f,g,1,1) : A_f \otimes A_g \to A_{fg}$ to $P_{--+}(g,f,1,g^{-1}) : A_g \otimes A_f \to A_{fg}$. Thus they are obtained from each other by the permutation of the two tensor factors. Thus, this shows the commutativity of the following diagram:

$$\begin{tikzcd}
A_f \otimes A_g \arrow[r, "\mu_{f,g}"] \arrow[d, "\tau"'] & A_{fg} \\
A_g \otimes A_f \arrow[r, "\phi_f \otimes 1"'] & A_{fgf^{-1}} \otimes A_f \arrow[u, "\mu_{fgf^{-1},g}"']
\end{tikzcd}$$

This topologically can be interpreted as the $\mathcal{X}$-homeomorphism between the two cobordisms below: The arrow in the above figure implies that the $\mathcal{X}$-morphism on

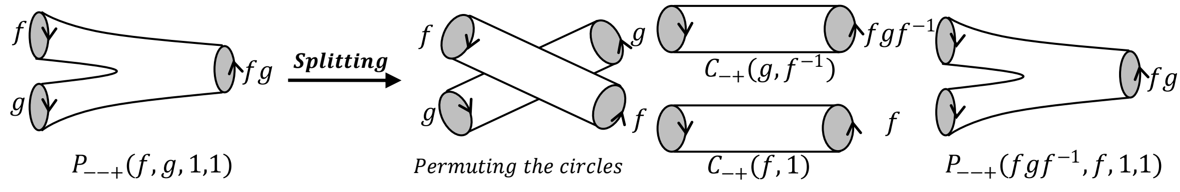



Figure 3.12: Topological interpretation of the above commutative diagram

left handside of the arrow is obtained from the composition of $\mathcal{X}$–morphisms on the right.

(vii) Trace condition. For any $f,g \in G$, let $A_f = (S^1_-, f)$ and $A_g = (S^1_-, g)$ be 0-morphisms in $\mathcal{X} - \mathrm{Cob}_1$. Let $h = fgf^{-1}g^{-1}$ and $A_h = (S^1_-, h)$. Consider the two following compositions of $\mathcal{X}$-morphisms: Hence,

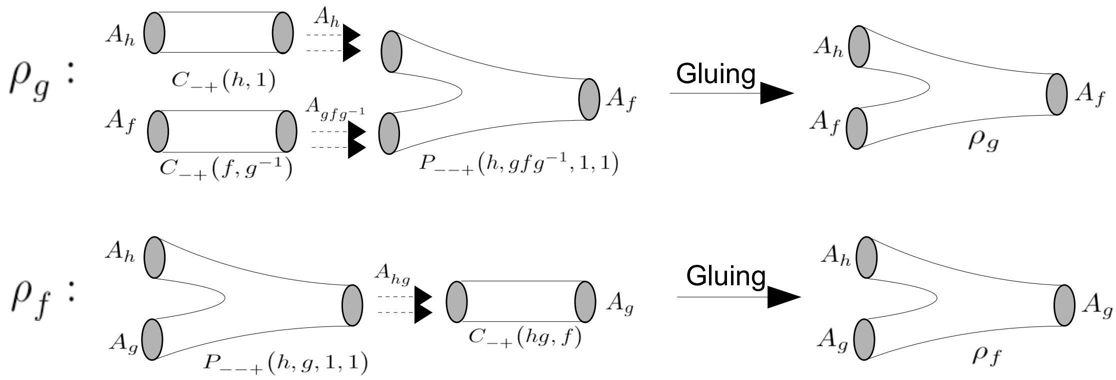



Figure 3.13: Morphisms for trace axiom

$$\rho_f = \mu_{fgf^1g^1,gfg^{-1}}(1 \otimes \varphi_g) : A_h \otimes A_f \to A_f$$

$$\rho_g = (\varphi_{f^{-1}})\mu_{fgf^1g^1,g} : A_h \otimes A_g \to A_g.$$

The trace axiom requires that the above two morphisms in $\mathcal{X} - \mathrm{Cob}_1$ have the same trace. Thus if

$$\mathrm{Tr}_X : \mathrm{Hom}\,(A_h \otimes X, X) \to \mathrm{Hom}\,(A_h, I)$$

is the trace morphism related to 0-morphism $X$ in $\mathcal{X} - \mathrm{Cob}_1$, then we need to show that

$$\mathrm{Tr}_{A_f}(\rho_f) = \mathrm{Tr}_{A_g}(\rho_g)$$

where each side is a morphism in $\mathcal{X} - \mathrm{Cob}_1$ from $A_h$ to $I$ given by figure (3.14). We

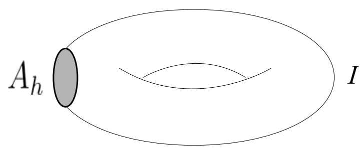



Figure 3.14: Trace Axiom

now construct a morphism $\mathcal{H} = (H, F, \alpha_H)$ in $\mathcal{X} - \mathrm{Cob}_1$ from $A_h$ to $I$ such that $\mathcal{H}$ will be the partial trace for $\rho_f$ as well as $\rho_g$.

Consider $S^1_-$ and fix one of its point, say $s$. Consider the 2-torus $S^1_- \times S^1_-$ with product orientation. Let $B \subset S^1_- \times S^1_-$ be a closed embedded 2-disc disjoint from the loops $S^1_- \times s$ and $s \times S^1_-$ for $s \in S^1_-$. Consider the punctured torus $H = (S^1_- \times S^1_-)/\operatorname{Int} B$ with orientation induced from $S^1_- \times S^1_-$. Let us provide the boundary circle $\partial H = \partial B$ with orientation opposite to the one induced from $H$. We choose a base point on $\partial H$ and an arc $r \subset H$ joining this point to $s \times s \in H$. We can assume that $r$ meets the loops $S^1_- \times s$ and $s \times S^1_-$ only in its endpoint $s \times s$.

Consider a map $F : H \to X = K(G, 1)$ such that $F(r) = x \in X$ and the restrictions of $F$ to $S^1_- \times s$, $s \times S^1_-$ represent $f$, $g \in G$, respectively. (Note that the orientations of $S^1_- \times s$, $s \times S^1_-$ are induced by the one of $S^1_-$.) Then the loop $F|_{\partial H}$ represents $fgf^{-1}g^{-1}$. Now, $\mathcal{H} = (H, F, \alpha_H)$ is 1-morphism from $(\partial H_-, F|_{\partial H}) = (A_h, h)$ to $I$. Thus we have a morphism, $\mathcal{H} : A_h \to I$.

Now $\mathcal{H}$ can be obtained from $P_{--+}(fgf^{-1}g^{-1}, f, 1, g)$ by gluing the boundary components $(M_-, f)$ and $(N_+, f)$ along an $\mathcal{X}$-homeomorphism. (Note that the circles $M_-$ and $N_+$ give the loop $S^1 \times s \subset N$). A standard argument in the theory of TQFTs shows that the homomorphism $\mathcal{H} : A_{fgf^{-1}g^{-1}} \to I$ is the partial trace of the homomorphism

$$\rho_f = P_{--+}(fgf^{-1}g^{-1}, f, 1, g) : A_{fgf^{-1}g^{-1}} \otimes A_f \to A_f.$$

Similarly, $\mathcal{H}$ is obtained from $P_{--+}(fgf^{-1}g^{-1}, g, f^{-1}, f^{-1})$ by gluing the boundary components $(M_-, g)$ and $(N_+, g)$ along an $\mathcal{X}$-homeomorphism. (Note that the circles $M_-$ and $N_+$ give the loop $s \times S^1 \in N$.) Thus, $\mathcal{H} : A_{fgf^{-1}g^{-1}, f, 1, g} \to I$ is the partial trace of the homomorphism

$$\rho_g = P_{--+}(fgf^{-1}g^{-1}, g, f^{-1}, f^{-1}) : A_{fgf^{-1}g^{-1}} \otimes A_g \to A_g.$$

□

Suppose $\mathcal{X}$ is a $K(G; 1)$ space. We can extend the above result to formulate the data given by any $\mathcal{X}$-HQFT with values in a monoidal category $\mathcal{C}$ to define a Turaev $G$-crossed system in $\mathcal{C}$. Suppose an $\mathcal{X}$-HQFT $(Z, \tau)$ with values in $\mathcal{C}$ be given. Then instead of working with circles and cylinders in $\mathcal{X} - \mathrm{Cob}_1$, we rather can work with the corresponding objects and morphisms in $\mathcal{C}$ given by $(Z, \tau)$.

Observe first that a 1-dimensional connected $\mathcal{X}$-manifold $\mathcal{M}$ is just a pointed oriented circle endowed with a map into $\mathcal{X}$ sending the base point into $x$. This is nothing but a loop in $\mathcal{X}$ with endpoints in $x$. If $Z_{\mathcal{M}}$ is the object in $\mathcal{C}$ given by $(Z, \tau)$ corresponding to $\mathcal{M}$, then clearly it depends only on the class of the loop in $\pi_1(\mathcal{X}) = G$. Thus for each $g \in G$, we obtain an object $A_g$ in $\mathcal{C}$.

Recall from Section (3.4.3), where we set our notation as $A_g$ for $(S^1_-; g)$. Instead (or, by abuse of notation), we have now set $A_g$ as an object in $\mathcal{C}$ given by a $\mathcal{X}$-HQFT $(Z, \tau)$ which corresponds to a 1-dimensional connected $\mathcal{X}$-manifold $\mathcal{M}$ which depends only on the class $g$ of the loop in $G$. Thus we have a collection $A = \{A_g\}$ of objects in $\mathcal{C}$. The $\mathcal{X}$-HQFT sends $\mathcal{X}$-cobordisms to morphisms in $\mathcal{C}$. Then, as in equations (3.11) to (3.14), we have the morphisms

$$\mu_{f,g} : A_f \otimes A_g \to A_{fg}$$

$$\eta : I_{\mathcal{C}} \to A_1$$

$$\epsilon : A_1 \to I_{\mathcal{C}}$$

$$\phi_f : A_f \otimes A_{f^{-1}} \to I$$

in $\mathcal{C}$ which equips the collection $A = \{A_g\}$ with a Frobenius structure (Proposition 3.4.3). Further, the calculations done in Theorem (3.4.4) are exactly the same so as to endow the Frobenius system $A = \{A_g\}$ with a Turaev structure. Thus the $\mathcal{X}$-HQFT $(Z, \tau)$ defines a Turaev $G$-crossed system $A = \{A_g\}$ in $\mathcal{C}$. We can summarise this discussion in the form of the following result:

**Theorem 3.4.6** *Suppose $\mathcal{X}$ is a $K(G; 1)$ space. Then any (1+1)-dimensional $\mathcal{X}$-HQFT with values in $\mathcal{C}$ defines a Turaev $G$-crossed system in $\mathcal{C}$.*

## 3.5 HQFTs

Assume $(S, \otimes, I)$ to be a symmetric monoidal category. Consider a Turaev crossed system $A = (A_g, \mu, \eta, \phi, \varphi)$ in $S$. Let $(Z, \tau)$ be a (1+1)-dimensional HQFT over $K(G, 1)$ space with values in $S$. The aim of this section is to reconstruct $(Z, \tau)$(at least up to isomorphism) from an underlying Turaev crossed system $A$. Throughout this section we shall use the term $\mathcal{X}$-surface for the base space of any $\mathcal{X}$-morphism in $\mathcal{X} - \mathrm{Cob}_1$.

### 3.5.1 Computing $\tau$

In this section we compute the morphisms (values of $\tau$) for (1+1)-dimensional $\mathcal{X}$-HQFT $(Z, \tau)$ when the underlying Turaev $G$-crossed system $A$ is given to us.

It follows from the topological classification of surfaces that every compact oriented surface can be split along a finite set of disjoint simple loops into a union of discs with $\leq 2$ holes, i.e., discs, annuli and discs with two holes. This implies that every $\mathcal{X}$-surface $W$ can be obtained by gluing from a finite collection of $\mathcal{X}$-surfaces which are discs with $\leq 2$ holes. Axioms of an HQFT imply that morphism $\tau(W)$ in $S$ is determined by the values of $\tau$ on the discs with $\leq 2$ holes. We show that these values are completely determined by $A$.

We begin by computing $\tau$ for $\mathcal{X}$-annuli. Each $\mathcal{X}$-annulus is isomorphic to one of these:

- $C_{-+}(g,h)$ or $C_{+-}(g,h)$
- $C_{--}(g,h)$
- $C_{++}(g,h)$

We define,

$$\tau\big(C_{-+}(g,h)\big) := \varphi_{g,h^{-1}} : A_g \to A_{h^{-1}gh} \quad \text{and} \tag{3.15}$$

$$\tau\big(C_{--}(g,1)\big) := \phi_g : A_g \otimes A_{g^{-1}} \to I.$$

The $\mathcal{X}$-annulus $C_{--}(g,h)$ can be obtained by the gluing of two $\mathcal{X}$-annuli $C_{-+}(g,h)$ and $C_{--}(h^{-1}gh, 1)$ along $(S^1_+, h^{-1}gh) = (S^1_-, h^{-1}gh)$. Algebraically, we have the following composition of morphisms in the category $S$:

$$A_g \otimes A_{h^{-1}g^{-1}h} \xrightarrow{\varphi_{g,h^{-1}} \otimes 1} A_{h^{-1}gh} \otimes A_{h^{-1}g^{-1}h} \xrightarrow{\phi_{h^{-1}gh}} I.$$

which helps us to define:

$$\tau(C_{--}(g;h) = \phi_{h^{-1}gh}(\varphi_{g,h^{-1}} \otimes 1) : A_g \otimes A_{h^{-1}g^{-1}h} \to I. \tag{3.16}$$

Note that here we make use of the gluing process in $\mathcal{X} - \mathrm{Cob}_1$ which corresponds to composition of morphisms in a category. Let us denote $\tau(C_{++}(g,h))$ as $F_{g,h}$. We require to define a morphism $F_{g,h} : I \to A_g \otimes A_{h^{-1}g^{-1}h}$. Consider the topological figure (3.15).

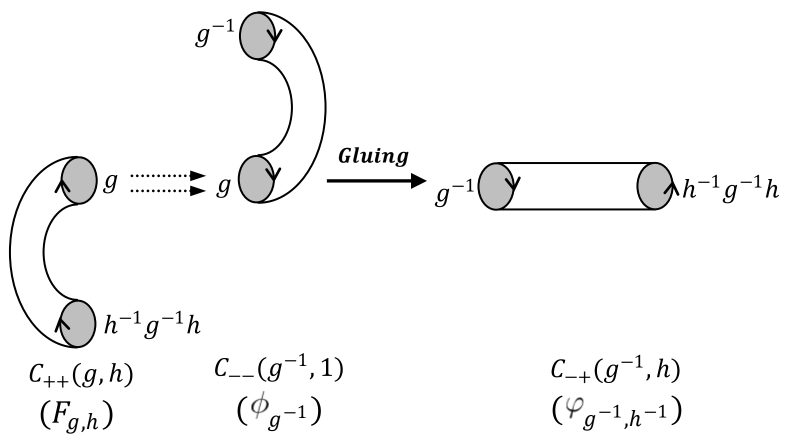



Figure 3.15: Morphism associated to $C_{++}(g,h)$.

We interpret this figure as the commutativity of the following algebraic diagram:

$$\begin{tikzcd}
A_{g^{-1}} = A_{g^{-1}} \otimes I \arrow[r, "1\otimes F_{g,h}"] \arrow[ddr, "\varphi_{g^{-1},h^{-1}}"'] & A_{g^{-1}} \otimes A_g \otimes A_{h^{-1}g^{-1}h} \arrow[d, "\phi_{g^{-1}}\otimes 1"] \\
& I \otimes A_{hgh^{-1}} \arrow[d, equal] \\
& A_{h^{-1}g^{-1}h}
\end{tikzcd}$$

i.e.

$$(\phi_{g^{-1}} \otimes 1) \circ (1 \otimes F_{g,h}) = \varphi_{g^{-1},h^{-1}}. \tag{3.17}$$

Then the gluing process in $\mathcal{X} - \mathrm{Cob}_1$ together with the non-degeneracy of the form $\phi$ uniquely defines and determines $F_{g,h}$. One can also consider it as the composition of the following maps:

$$F_{g,h} : I \xrightarrow{\eta} A_e \xrightarrow{\Delta_{g,g^{-1}}} A_g \otimes A_{g^{-1}} \xrightarrow{1\otimes\varphi_{g^{-1},h^{-1}}} A_g \otimes A_{h^{-1}g^{-1}h}.$$

where $\Delta = \{\Delta_{f,g} : A_{fg} \to A_f \otimes A_g; f,g \in G\}$ is one of the structures of $A$ which equips it with a Frobenius $G$-graded system.

There are two $\mathcal{X}$ discs : $D_+$ where the orientation of the boundary is induced by the one in disc and $D_-$ where the orientation of the boundary is opposite to the one induced from the disc. We set,

$$\tau(D_+) = \eta : I \to A_e.$$

The $\mathcal{X}$ disc $D_-$ may be obtained by the gluing of $D_+$ and $C_{--}(1;1)$ along $\partial D_+ = (S^1_+;1) \cong_{\mathcal{X}} (C^0_-,1)$. Topologically,

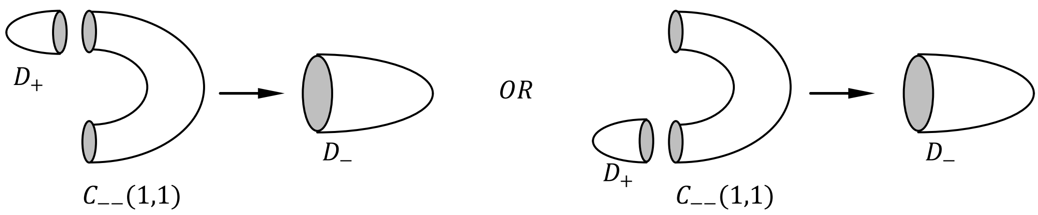



Figure 3.16: Disc $D_-$

This gives corresponding algebraic picture:

$$\begin{tikzcd}
A_e \arrow[r, equal] \arrow[dr, equal] & A_e \otimes I \arrow[r, "(1\otimes\eta)"] & A_e \otimes A_e \arrow[r, "\phi_1"] & I \\
& I \otimes A_e \arrow[ur, "(\eta\otimes 1)"'] & &
\end{tikzcd}$$

Thus we define,

$$\tau(D_-) = \phi_1(1 \otimes \eta) = \phi_1(\eta \otimes 1).$$

Let us now consider a $(2,\mathcal{X})$-pant in $\mathcal{X} - \mathrm{Cob}_1$. Each $(2,\mathcal{X})$-pant $P$ can be split as a union of two $\mathcal{X}$-annuli and a smaller $\mathcal{X}$-pant. Choosing appropriate orientations of loops we obtain that any $\mathcal{X}$-pant $P_{\epsilon,\mu,\delta}(g_1,g_2,g_3,g_4)$ splits as a union of two $\mathcal{X}$-annuli and an $\mathcal{X}$-pant isomorphic to $P_{--+}(g,h,1,1)$ for $g,h \in G$. Below we show the construction of $P_{--+}(g_1,g_2,g_3,g_4)$ :

Thus we first need to set and define $\tau\big(P_{--+}(g,h,1,1)\big)$. Let us see what does this figure

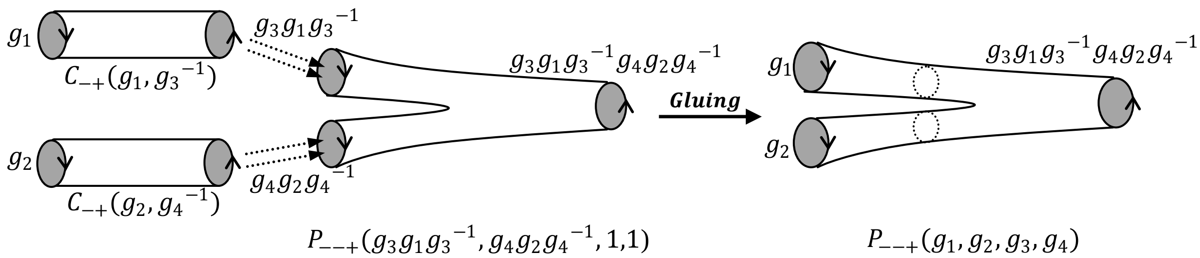



Figure 3.17: Morphism associated to $P_{--+}(g_1, g_2, g_3, g_4)$

imply algebraically. It essentially requires the commutativity of the following diagram:

$$\begin{tikzcd}
A_{g_1} \otimes A_{g_2} \arrow[r, "\varphi_{g_1,g_3} \otimes \varphi_{g_2,g_4}"] \arrow[dr] & A_{g_3g_1g_3^{-1}} \otimes A_{g_4g_2g_4^{-1}} \arrow[d] \\
& A_{g_3g_1g_3^{-1}g_4g_2g_4^{-1}}
\end{tikzcd}$$

Note that the map $A_{g_3g_1g_3^{-1}} \otimes A_{g_4g_2g_4^{-1}} \to A_{g_3g_1g_3^{-1}g_4g_2g_4^{-1}}$ in the above diagram corresponds to $\tau\big(P_{--+}\big)(g_3g_1g_3^{-1}, g_4g_2g_4^{-1}, 1, 1)$. Thus let us define

$$\tau(P_{--+})(g, h, 1, 1) = \mu_{g,h} : A_g \otimes A_h \to A_{gh}.$$

We can now define $\tau(P_{--+})$ as follows:

$$\tau\big(P_{--+}\big)(g_1, g_2, g_3, g_4) = \mu_{g_3g_1g_3^{-1}, g_4g_2g_4^{-1}}(\varphi_{g_1,g_3} \otimes \varphi_{g_2,g_4}). \tag{3.18}$$

Note that here we are again making use of the gluing process in $\mathcal{X} - \mathrm{Cob}_1$.

Now consider the $(2, \mathcal{X})$-pant, $P = P_{---}(g_1, g_2, g_3, g_4)$. Let us set

$$g = g_4g_2^{-1}g_4^{-1}g_3g_1^{-1}g_3^{-1}.$$

Then,

$$\tau(P) = \phi_{g^{-1}}(\mu_{g_3g_1g_3^{-1}, g_4g_2g_4^{-1}} \otimes 1_g)(\varphi_{g_1,g_3} \otimes \varphi_{g_2,g_4} \otimes 1_g). \tag{3.19}$$

This definition of $P$ follows immediately as a result of gluing of $P_{--+}(g_1, g_2, g_3, g_4)$ to

$C_{--}(g^{-1};1)$ along $(N_+, g^{-1}) \cong_{\mathcal{X}} (S^1_-; g^{-1}) = C^0_-(g^{-1})$.

Consider the $(2,\mathcal{X})$-pant, $P = P_{++-}(g_1, g_2, g_3, g_4)$. We can obtain $P$ by gluing three cylinders $C_{++}(g_1, 1)$, $C_{++}(g_2, 1)$, $C_{--}(g, 1)$, $g$ as fixed before, to the pant $P_{--+}(g_1^{-1}, g_2^{-1}, g_3, g_4)$ along $\mathcal{X}$-homeomorphisms

$$(L_-, g_1^{-1}) \cong_{\mathcal{X}} (S^1_+; g_1^{-1}) = (C^1_+, g_1^{-1}),$$

$$(M_-, g_2^{-1}) \cong_{\mathcal{X}} (S^1_+; g_2^{-1}) = (C^1_+, g_2^{-1}),$$

$$(N_+, g) \cong_{\mathcal{X}} (S^1_+; g) = (C^0_-, g).$$

respectively. Following diagram shows the construction:

$$\begin{tikzcd}
A_{g^{-1}} = A_{g^{-1}} \otimes I \otimes I \arrow[r, "1\otimes\tau(C_{++})\otimes\tau(C_{++})"] & A_{g^{-1}} \otimes A_{g_1} \otimes A_{g_1^{-1}} \otimes A_{g_2} \otimes A_{g_2^{-1}} \arrow[d, "1\otimes1\otimes1\otimes\tau"] \\
& A_{g^{-1}} \otimes A_{g_1} \otimes A_{g_1^{-1}} \otimes A_{g_2^{-1}} \otimes A_{g_2} \arrow[d, "1\otimes1\otimes\tau(P_{--+})\otimes1"] \\
& A_{g^{-1}} \otimes A_{g_1} \otimes A_g \otimes A_{g_2} \arrow[d, "\tau\otimes1\otimes1"] \\
& A_{g_1} \otimes A_{g^{-1}} \otimes A_g \otimes A_{g_2} \arrow[d, "1\otimes\tau(C_{--})\otimes1"] \\
& A_{g_1} \otimes A_{g_2}.
\end{tikzcd}$$

Similarly, we can obtain $P_{+++}(g_1, g_2, g_3, g_4)$ by gluing 3 cylinders of type $C_{++}(*, 1)$ to $P_{---}(g_1^{-1}, g_2^{-1}, g_3, g_4)$. By the gluing process in $\mathcal{X} - \mathrm{Cob}_1$, it will be the composition of the maps given below:

$$\begin{tikzcd}
I \arrow[r, "\tau(C_{++}(g_1;1))\otimes\tau(C_{++}(g_2;1))\otimes\tau(C_{++}(g;1))"] & A_{g_1}\otimes A_{g_1^{-1}}\otimes A_{g_2}\otimes A_{g_2^{-1}}\otimes A_g\otimes A_{g^{-1}} \arrow[d, "1\otimes\tau\otimes1\otimes\tau"] \\
& A_{g_1}\otimes A_{g_2}\otimes A_{g_1^{-1}}\otimes A_{g_2^{-1}}\otimes A_{g^{-1}}\otimes A_g \arrow[d, "1\otimes1\otimes\tau(P_{---})\otimes1"] \\
& A_{g_1}\otimes A_{g_2}\otimes A_g,
\end{tikzcd}$$

where $g = g_3 g_1^{-1} g_3^{-1} g_4 g_2^{-1} g_4^{-1}$. Further, for $P_{-++}(g_1, g_2, g_3, g_4)$ we glue a cylinder of type $C_{--}(g_1, 1)$ to a pant of type $P_{+++}(g_1^{-1}, g_2, g_3, g_4)$. It is given by the following composition:

$$\tau(P_{-++}(g_1, g_2, g_3, g_4)) = \big[\tau(C_{--}(g_1;1))\otimes1\otimes1\big]\circ\big[1\otimes\tau(P_{+++}(g_1^{-1}, g_2, g_3, g_4))\big] : A_{g_1}\to A_{g_2}\otimes A_g; \tag{3.20}$$

where $g = g_3 g_1 g_3^{-1} g_4 g_2^{-1} g_4^{-1}$. Similarly, we can obtain $P_{++-}(g_1, g_2, g_3, g_4)$ by gluing 2 cylinders of type $C_{++}(*, 1)$ to $P_{---}(g_1^{-1}, g_2^{-1}, g_3, g_4)$. The composition of the following maps below gives the corresponding morphism for $P_{++-}$:

$$\begin{tikzcd}
I\otimes A_g \arrow[r, "\tau(C_{++}(g_1;1))\otimes\tau(C_{++}(g_2;1))\otimes1"] & (A_{g_1}\otimes A_{g_1^{-1}})\otimes(A_{g_2}\otimes A_{g_2^{-1}})\otimes A_g \arrow[d, "1\otimes\tau\otimes1"] \\
& A_{g_1}\otimes A_{g_2}\otimes A_{g_1^{-1}}\otimes A_{g_2^{-1}}\otimes A_g \arrow[d, "1\otimes1\otimes\tau(P_{---})"] \\
& A_{g_1}\otimes A_{g_2},
\end{tikzcd}$$

where $g = g_4 g_2^{-1} g_4^{-1} g_3 g_1^{-1} g_3^{-1}$.

Finally we define $\tau(W)$ for any connected $\mathcal{X}$-surface $(W, p : W \to X)$. By a splitting system of loops on $W$ we mean a finite set of disjoint embedded circles $\alpha_1, \cdots, \alpha_N \subset W$ which split $W$ into a union of discs with $\leq 2$ holes. We provide each $\alpha_i$ with an orientation and a base point $x_i$ such that $p(x_i) = x \in X$ for all $i$. The discs with holes obtained by the splitting of $W$ along $\cup_i \alpha_i$ endowed with the restriction of $p$ are $\mathcal{X}$-surfaces. The gluing process in $\mathcal{X} - \mathrm{Cob}_1$ determines $\tau(W)$ from the values of $\tau$ on these discs with holes. Thus, we have defined $\tau$ for any connected $\mathcal{X}$-surface. We can then extend $\tau$ to any arbitrary $\mathcal{X}$-surface using the monoidal structure of $\mathcal{X} - \mathrm{Cob}_1$.

**Lemma 3.5.1** *The morphisms in $S$ defined in equations (3.15) to (3.20) which corresponds to $\mathcal{X}$-cobordisms are all well defined.*

PROOF: For the well defineness of the morphisms we need to show the topological invariance of the morphisms under $\mathcal{X}$-homeomorphisms. The $\mathcal{X}$-homeomorphisms of $(2, \mathcal{X})$-cylinders are generated by (1) the Dehn twists $C_{\epsilon,\mu}(g;h) \to C_{\epsilon,\mu}(g;hg)$ along the circle $S^1 \times (1/2)$ where $\epsilon, \mu = \pm$ and (2) the $\mathcal{X}$-homeomorphisms $C_{\epsilon,\epsilon}(g,h) \to C_{\epsilon,\epsilon}(h^{-1}g^{-1}h, h^{-1})$ permuting the boundary components of the cylinder and preserving the arc $s \times [0,1]$ (with $s \in S^1$) as a set. So, to establish the topological invariance for $(2, \mathcal{X})$-cylinders, it is suffices to check the invariance under these two morphisms.

The invariance of $\tau$ under the Dehn twists for $C_{-+}$ is as follows:

$$\tau(C_{-+}(f, h^{-1})) = \varphi_{f,h} : A_f \to A_{hfh^{-1}}$$

$$\tau(C_{-+}(f, (hg)^{-1})) = \varphi_{f,hg} : A_f \to A_{hfh^{-1}}.$$

Now, $\varphi_{g,hg} = (\varphi_{g,h} \circ \varphi_{g,g}) = \varphi_{g,h}$. The following calculation shows the invariance of $\tau$ under Dehn twists for $C_{--}$ :

$$\begin{aligned}
\tau\big(C_{--}(g, hg)\big) &= \phi_{(hg)g(hg)^{-1}}\big(\varphi_{g,hg} \otimes 1\big) \\
&= \phi_{hgh^{-1}}\big(\varphi_{g,h} \otimes 1\big) \\
&= \tau\big(C_{--}(g, h)\big).
\end{aligned}$$

The morphism $\tau$ defined for $C_{--}$ is invariant under the homeomorphism (2) as follows

$$\begin{aligned}
\tau(C_{--}(g,h)) &= \phi_{h^{-1}gh}(\varphi_{g,h^{-1}} \otimes 1_{h^{-1}g^{-1}h}) \\
&= \phi_{h^{-1}gh}(\varphi_{g,h^{-1}} \otimes \varphi_{h^{-1}g^{-1}h,1}) \\
&= \phi_{h^{-1}gh}(\varphi_{g,h^{-1}} \otimes \varphi_{g^{-1},h^{-1}}\varphi_{h^{-1}g^{-1}h,h}) \\
&= \phi_{g}(1_g \otimes \varphi_{h^{-1}g^{-1}h,h})
\end{aligned}$$

$$\begin{aligned}
\tau(C_{--}(h^{-1}g^{-1}h,h^{-1})) &= \phi_{g^{-1}}(\varphi_{h^{-1}g^{-1}h,h} \otimes 1_g) \\
&= \phi_{g}(1_g \otimes \varphi_{h^{-1}g^{-1}h,h}).
\end{aligned}$$

Here we have used the invariance of $\phi$ under $\varphi$ (Axiom 3.3), and that $\phi$ is a symmetric form. To prove that the morphism $\tau(C_{++}(g,h))$ is invariant under the homeomorphism (2), consider the following observation:
By definition (from (3.17)):

$$(\phi_{g^{-1}} \otimes 1)(1 \otimes F_{g,h}) = \varphi_{g^{-1},h^{-1}}.$$

$$\text{and,} \quad (\phi_{h^{-1}gh} \otimes 1)(1 \otimes F_{h^{-1}g^{-1}h,h^{-1}}) = \varphi_{h^{-1}gh,h}. \tag{3.21}$$

Moreover, the multiplicativity of $\varphi$ implies

$$\varphi_{h^{-1}g^{-1}h,h} \cdot \varphi_{g^{-1},h^{-1}} = \varphi_{g^{-1},1} = 1 = \text{identity map}.$$

Crossing preserves the form $\big($Axiom (3.3)$\big)$ gives: $\phi_{g^{-1}}(1 \otimes \varphi_{h^{-1}gh,h})$

$$\begin{aligned}
&= \phi_{g^{-1}}\big(\varphi_{h^{-1}g^{-1}h,h} \cdot \varphi_{g^{-1},h^{-1}} \otimes \varphi_{h^{-1}gh,h}\big) \\
&= \phi_{h^{-1}g^{-1}h}\big(\varphi_{g^{-1},h^{-1}} \otimes 1\big) \\
&= \phi_{h^{-1}g^{-1}h}\big\{(\phi_{g^{-1}} \otimes 1) \circ (1 \otimes F_{g,h}) \otimes 1\big\} \\
&= \phi_{h^{-1}gh}\tau\big\{(\phi_{g^{-1}} \otimes 1) \circ (1 \otimes F_{g,h}) \otimes 1\big\} \\
&= \phi_{h^{-1}gh}\big\{1 \otimes (\phi_{g^{-1}} \otimes 1) \circ (1 \otimes F_{g,h})\big\}.
\end{aligned}$$

Using (3.17):

$$\phi_{g^{-1}}(1 \otimes \varphi_{h^{-1}gh,h}) = \phi_{g^{-1}}\Big[1 \otimes (\phi_{h^{-1}gh} \otimes 1)(1 \otimes F_{h^{-1}g^{-1}h,h^{-1}})\Big].$$

Thus, we get:

$$\phi_{g^{-1}}\Big[1 \otimes (\phi_{h^{-1}gh} \otimes 1)(1 \otimes F_{h^{-1}g^{-1}h,h^{-1}})\Big] = \phi_{h^{-1}gh}\Big[1 \otimes (\phi_{g^{-1}} \otimes 1)(1 \otimes F_{g,h})\Big].$$

Finally, the non-degeneracy of the form $\phi$ implies:

$$F_{h^{-1}g^{-1}h,h^{-1}} = F_{g,h}.$$

Thus, $\tau\big(C_{++}(h^{-1}g^{-1}h, h^{-1})\big) = \tau\big(C_{++}(g,h)\big)$.

For an $(2, \mathcal{X})$-pant $P = P_{--+}(g_1, g_2, g_3, g_4)$, any self homeomorphism is isotopic to a composition of Dehn twists in annuli neighbourhoods of circles $L, M \subset \partial P$ and the self homeomorphisms $f^{\pm} : P \to P$ which is identity on $N$ and permutes the circles in the inboundary, i.e. $L$ and $M$. An easy computation shows that $f$ is an $\mathcal{X}$-homeomorphism from $P_{--+}(g_1, g_2, 1, 1)$ to $P_{--+}(g_2, g_1, 1, g_2^{-1})$. Now,

$$\tau(P_{--+})(g_1, g_2, 1, 1)) : A_{g_1} \otimes A_{g_2} \to A_{g_1 g_2}$$

$$\tau(P_{--+})(g_2, g_1, 1, g_2^{-1})) : A_{g_2} \otimes A_{g_1} \to A_{g_2 g_1}$$

where $\tau(P_{--+})(g_1, g_2, 1, 1)) = \mu_{g_1,g_2}(\varphi_{g_1,1} \otimes \varphi_{g_2,1}) = \mu_{g_1,g_2}$ and $\tau(P_{--+})(g_2, g_1, 1, g_2^{-1})) = \mu_{g_2,g_2^{-1}g_1g_2}(\varphi_{g_2,1} \otimes \varphi_{g_1,g_2^{-1}}) = \mu_{g_2,g_2^{-1}g_1g_2}(1_{g_2} \otimes \varphi_{g_1,g_2^{-1}}) = \mu_{g_1,g_2}\tau_{A_{g_2},A_{g_1}}$, using axiom (3.5). Thus $\tau(P_{--+}(g_1, g_2, 1, 1))$ equals $\tau(P_{--+}(g_2, g_1, 1, g_2^{-1}))$ as $A_{g_1} \otimes A_{g_2} = A_{g_2} \otimes A_{g_1}$ in $\mathcal{X} - \mathrm{Cob}_1$.

For checking the invariance of $P = P_{---}(g_1, g_2, g_3, g_4)$, consider the $\mathcal{X}$-homeomorphism $f : P \to P$, which maps $(L, l)$, $(M, m)$, $(N, n)$ onto $(M, m)$, $(N, n)$, $(L, l)$, respectively. An easy computation shows that $f$ is an $\mathcal{X}$-homeomorphism from $P = P_{---}(g_1g_2, g_3, g_4)$ to $P' = P_{---}(g, g_1, g_4^{-1}, g_4^{-1}g_3)$. Then invariance of $\tau$ under $f$ is as follows:

$$\begin{aligned}
\tau(P') &= \phi_{g_2^{-1}}\Big\{\mu_{g_4^{-1}gg_4,g_4^{-1}g_3g_1g_3^{-1}g_4}\Big(\varphi_{g,g_4^{-1}} \otimes \varphi_{g_1,g_4^{-1}g_3} \otimes \varphi_{g_2}\Big) \otimes 1_{g_2}\Big\} \\
&= \phi_{g_4g_2^{-1}g_4^{-1}}\Big\{\varphi_{g_2^{-1},g_4}\Big[\mu_{g_4^{-1}gg_4,g_4^{-1}g_3g_1g_3^{-1}g_4}\Big(\varphi_{g,g_4^{-1}} \otimes \varphi_{g_1,g_4^{-1}g_3}\Big)\Big] \otimes \varphi_{g_2,g_4}(1_{g_1,g_4^{-1}g_3})\Big\} \\
&= \phi_{g_4g_2^{-1}g_4^{-1}}\Big\{\mu_{g,g_3g_1g_3^{-1}}\Big[\varphi_{g_4^{-1}gg_4,g_4} \cdot \varphi_{g,g_4^{-1}} \otimes \varphi_{g_4^{-1}g_3g_1g_3^{-1}g_4,g_4} \cdot \varphi_{g_1,g_4^{-1}g_3} \otimes \Big] \otimes \varphi_{g_2,g_4}\Big\} \\
&= \phi_{g_4g_2^{-1}g_4^{-1}}\Big\{\mu_{g,g_3g_1g_3^{-1}}\Big[\varphi_{g,g_4g_4^{-1}} \otimes \varphi_{g_1g_3}\Big] \otimes \varphi_{g_2,g_4}\Big\} \\
&= \phi_{g_4g_2^{-1}g_4^{-1}}\Big\{\mu_{g,g_3g_1g_3^{-1}}\Big[1_g \otimes \varphi_{g_1g_3}\Big] \otimes \varphi_{g_2,g_4}\Big\} \\
&= \phi_{g_4g_2^{-1}g_4^{-1}}\Big\{\Big(\mu_{g,g_3g_1g_3^{-1}} \otimes 1_{g_4g_2g_4^{-1}}\Big) \circ \Big(1_g \otimes \varphi_{g_1g_3} \otimes \varphi_{g_2g_4}\Big)\Big\} \\
&= \phi_g\Big\{\Big(1_g \otimes \mu_{g_3g_1g_3^{-1},g_4g_2g_4^{-1}}\Big) \circ \Big(1_g \otimes \varphi_{g_1g_3} \otimes \varphi_{g_2g_4}\Big)\Big\} \\
&= \phi_g\Big\{1_g \otimes \mu_{g_3g_1g_3^{-1},g_4g_2g_4^{-1}}\Big(\varphi_{g_1g_3} \otimes \varphi_{g_2g_4}\Big)\Big\} \\
&= \phi_{g^{-1}}\Big\{\mu_{g_3g_1g_3^{-1},g_4g_2g_4^{-1}}\Big(\varphi_{g_1g_3} \otimes \varphi_{g_2g_4}\Big) \otimes 1_g\Big\} \\
&= \tau(P).
\end{aligned}$$

Here we have intensively used axioms (3.2), (3.3), associativity of $\mu$ and symmetricity of $\phi$. Note that any self-$\mathcal{X}$-homeomorphism of $P$ is a composition of $f^{\pm}$ with a self-homeomorphism of $P$ preserving all boundary components set-wise. Thus it remains only to check the topological invariance of $\tau(P)$ under self-homeomorphisms of $P$ preserving the boundary components. Such homeomorphisms of $P$ (considered up to isotopy) are compositions of Dehn twists in annuli neighbourhoods of $L$ , $M$, $N$ . The invariance of $\tau(P)$ under such Dehn twists follows from the already established topological invariance of the morphisms $\tau(P_{--+}(g_1,g_2,g_3,g_4))$ and $\tau\big(C_{--}(g^{-1},1)\big)$.

The topological invariance of $P = P_{++-}(g_1,g_2,g_3,g_4)$ follows from the topological invariance of the values of $\tau$ for $P_{--+}(g_1^{-1},g_2^{-1},g_3,g_4)$ and the three $(2,\mathcal{X})$-annuli : $C_{++}(g_1,1)$, $C_{++}(g_2,1)$, $C_{--}(g,1)$ and the following fact: any self-homeomorphism of $P$ is isotopic to a homeomorphism mapping a given neighbourhood of $\partial P$ onto itself. thus we have checked the topological invariance of all the morphisms defined before the lemma. This completes the proof. $\square$

Before the next lemma, we discuss the multiplicativity of $\tau$ which is as follows: Let $g_1,g_2,g_3,\ g_4,h,k \in G$. Set $g = g_3g_1g_3^{-1}g_4g_2g_4^{-1}$. Observe that the gluing of $P_{--+}$ to $C_{--}(g,h)$ along $(N_+,g) \cong (C_-^0,g)$ yields the $\mathcal{X}$-pant : $P = P_{---}(g_1,g_2,h^{-1}g_3,h^{-1}g_4)$. The same $\mathcal{X}$-pant $P$ is also obtained by gluing $\mathcal{X}$-pant $\widetilde{P} = P_{-+-}(g_1,k^{-1}g_2k,h^{-1}g_3,h^{-1}g_4))$ to $C_{--}(g_2,k)$ along $(M_+,k^{-1}g_2^{-1}k) \cong (C_-^1,k^{-1}g_2^{-1}k)$. This allows us to compute $\tau(P)$, using the gluing process in $\mathcal{X}-\mathrm{Cob}_1$ to these two splittings of $P$. We claim that these two computations give the same result. The first splitting gives:

$$\begin{aligned}
\tau(P) &= \tau\big(C_{--}(g,h)\big)\Big[\tau\big(P_{--+}(g_1,g_2,g_3,g_4)\big)\otimes C_{-+}(h^{-1}gh,1)\Big]\\
&= \phi_{h^{-1}gh}(\varphi_{g,h^{-1}}\otimes 1)\big(\mu_{g_3g_1g_3^{-1},g_4g_2g_4^{-1}}\otimes 1\big)\big(\varphi_{g_1,g_3}\otimes\varphi_{g_2,g_4}\otimes 1_{h^{-1}gh}\big)\\
&= \phi_{h^{-1}gh}\Big[\varphi_{g,h^{-1}}\mu_{g_3g_1g_3^{-1},g_4g_2g_4^{-1}}\big(\varphi_{g_1,g_3}\otimes\varphi_{g_2,g_4}\big)\otimes 1_{h^{-1}gh}\Big]\\
&= \phi_{h^{-1}gh}\Big[\mu_{h^{-1}g_3g_1g_3^{-1}h,{}^{-1}g_4g_2g_4^{-1}h}\big(\varphi_{g_3g_1g_3^{-1},h^{-1}}\varphi_{g_1,g_3}\otimes\varphi_{g_4g_2g_4^{-1},h^{-1}}\varphi_{g_2,g_4}\big)\\
&\qquad\otimes 1_{h^{-1}gh}\Big]\ \text{, by axiom (3.2)}\\
&= \phi_{h^{-1}gh}\Big[\mu_{h^{-1}g_3g_1g_3^{-1}h,{}^{-1}g_4g_2g_4^{-1}h}\big(\varphi_{g_1,h^{-1}g_3}\otimes\varphi_{g_2,h^{-1}g_4}\big)\otimes 1_{h^{-1}gh}\Big]\ \text{, by axiom (3.1).}
\end{aligned}$$

To use the second splitting we first observe that $\widetilde{P}$ is $\mathcal{X}$-homeomorphic to

$$P_{--+}(h^{-1}gh,g_1,k^{-1}g_4^{-1}h,k^{-1}g_4^{-1}g_3)$$

via an $\mathcal{X}$-homeomorphism mapping the boundary components $L$, $M$, $N$ onto $M$, $N$, $L$, respectively. Therefore applying gluing axiom to the second splitting of $P$ we obtain the value of $\tau(P)$

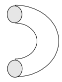
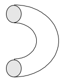
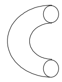
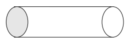
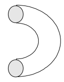
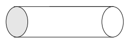
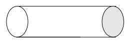
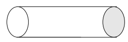
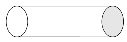
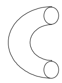
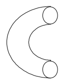
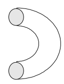
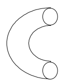
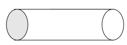
$$
\begin{aligned}
&= \tau(C_{--}(g_2,k)\Big[C_{-+}(k^{-1}g_2k,1)\otimes\tau\big(P_{--+}(h^{-1}gh,g_1,k^{-1}g_4^{-1}h,k^{-1}g_4^{-1}g_3)\big)\Big]\\
&= \phi_{k^{-1}g_2k}(\varphi_{g_2,k^{-1}}\otimes 1_{k^{-1}g_2^{-1}k})\big[1_{k^{-1}g_2k}\otimes\mu_{k^{-1}g_4^{-1}g^{-1}g_4k,k^{-1}g_4^{-1}g_3g_1g_3^{-1}g_4k}\\
&\quad\big(\varphi_{h^{-1}g^{-1}h,k^{-1}g_4^{-1}h}\otimes\varphi_{g_1,g_4^{-1}g_3}\big)\big]\\
&= \phi_{k^{-1}g_2k}\Big[\varphi_{g_2,k^{-1}}\otimes\mu_{k^{-1}g_4^{-1}g^{-1}g_4k,k^{-1}g_4^{-1}g_3g_1g_3^{-1}g_4k}\\
&\quad\big(\varphi_{g_4^{-1}g^{-1}g_4,k^{-1}}\varphi_{h^{-1}g^{-1}h,g_4^{-1}h}\otimes\varphi_{g_4^{-1}g_3g_1g_3^{-1}g_4,k^{-1}}\varphi_{g_1,g_4^{-1}g_3}\big)\Big]\\
&= \phi_{k^{-1}g_2k}\Big[\varphi_{g_2,k^{-1}}\otimes\varphi_{g_2^{-1},k^{-1}}\mu_{g_4^{-1}g^{-1}g_4,g_4^{-1}g_3g_1g_3^{-1}g_4}\\
&\quad\big(\varphi_{h^{-1}g^{-1}h,g_4^{-1}h}\otimes\varphi_{g_1,g_4^{-1}g_3}\big)\Big] \quad \text{, by axiom (3.2)}\\
&= \phi_{g_2}\Big[1_{g_2}\otimes\mu_{g_4^{-1}g^{-1}g_4,g_4^{-1}g_3g_1g_3^{-1}g_4}\big(\varphi_{h^{-1}g^{-1}h,g_4^{-1}h}\otimes\varphi_{g_1,g_4^{-1}g_3}\big)\Big] \quad \text{, by axiom (3.3)}\\
&= \phi_{g_2^{-1}}\Big[\mu_{g_4^{-1}g^{-1}g_4,g_4^{-1}g_3g_1g_3^{-1}g_4}\big(\varphi_{h^{-1}g^{-1}h,g_4^{-1}h}\otimes\varphi_{g_1,g_4^{-1}g_3}\big)\otimes 1_{g_2}\Big] \quad \text{, by Lemma (3.4.1)(c)}\\
&= \phi_{g_4^{-1}g^{-1}g_4}\Big[\varphi_{h^{-1}g^{-1}h,g_4^{-1}h}\otimes\mu_{g_4^{-1}g_3g_1g_3^{-1}g_4,g_2}\big(\varphi_{g_1,g_4^{-1}g_3}\otimes 1_{g_2}\big)\Big] \quad \text{, by property of } \mu\\
&= \phi_{g_4^{-1}gg_4}\Big[\mu_{g_4^{-1}g_3g_1g_3^{-1}g_4,g_2}\big(\varphi_{g_1,g_4^{-1}g_3}\otimes 1_{g_2}\big)\otimes\varphi_{h^{-1}g^{-1}h,g_4^{-1}h}\Big] \quad \text{, by Lemma 3.4.1(c)}\\
&= \phi_{h^{-1}gh}\Big[\varphi_{g_4^{-1}gg_4,h^{-1}g_4,h^{-1}g_4}\ \mu_{g_4^{-1}g_3g_1g_3^{-1}g_4,g_2}\big(\varphi_{g_1,g_4^{-1}g_3}\otimes 1_{g_2}\big)\\
&\quad\otimes\big(\varphi_{g_4^{-1}g^{-1}g_4,h^{-1}g_4,h^{-1}g_4}\ \varphi_{h^{-1}g^{-1}h,g_4^{-1}h}\big)\Big] \quad \text{, by axiom (3.2)}\\
&= \phi_{h^{-1}gh}\Big[\mu_{h^{-1}g_3g_1g_3^{-1}h,h^{-1}g_4g_2g_4^{-1}h}\big(\varphi_{g_4^{-1}g_3g_1g_3^{-1}g_4,h^{-1}g_4}\ \varphi_{g_1,g_4^{-1}g_3}\otimes\varphi_{g_2,h^{-1}g_4}\big)\\
&\quad\otimes 1_{h^{-1}g^{-1}h}\Big] \quad \text{, by other side of axiom (3.2) and Lemma 3.4.1(a)}\\
&= \phi_{h^{-1}gh}\Big[\mu_{h^{-1}g_3g_1g_3^{-1}h,h^{-1}g_4g_2g_4^{-1}h}\big(\varphi_{g_1,h^{-1}g_3}\ \varphi_{g_2,h^{-1}g_4}\big)\\
&\quad\otimes 1_{h^{-1}g^{-1}h}\Big] \quad \text{, by axiom (3.1)}
\end{aligned}
$$

We have our next lemma before the main result of the section:

**Lemma 3.5.2** *The morphisms (3.15)-(3.20) in $S$ corresponding to $\mathcal{X}$-cobordisms satisfy gluing axiom in the definition of an HQFT.*

PROOF: We start the proof with the $(2,\mathcal{X})$-annulus. For any $(2,\mathcal{X})$-annuli it is suffices to consider the case where the annulus $C_{\epsilon,\mu}(g_1,g_2)$ is glued to $C_{-\mu,\nu}(g_3,g_4)$ along an $\mathcal{X}$-homeomorphism $\big(C^1_\mu,(g_2^{-1}g_1^{-\epsilon}g_2)^\mu\big)=\big(C^0_{-\mu},g_3\big)$. Note that the gluing is possible only if $g_3=(g_2^{-1}g_1^{-\epsilon}g_2)^\mu$ ; the result of the gluing is the $(2,\mathcal{X})$-cylinder $C_{\epsilon,\nu}(g_1,g_2g_4)$. Depending on the values of $\epsilon,\mu,\nu=\pm$, following eight cases arise, which are given in the table below.

We indicate the key argument implying the axiom (6) in all the possible eight cases. The table below shows all the cases with the shaded boundaries indicating the inboundary of the cylinders which are provided with negative orientation.

| | **ε** | **μ** | **ν** | **$C = C_{\varepsilon,\mu}(g_1,g_2)$** | **$C' = C_{-\mu,\nu}(g_3,g_4)$** |
|---|---|---|---|---|---|
| **1** | - | - | - |  |  |
| **2** | - | - | + |  |  |
| **3** | - | + | - |  |  |
| **4** | - | + | + |  |  |
| **5** | + | - | - |  |  |
| **6** | + | - | + |  |  |
| **7** | + | + | - |  |  |
| **8** | + | + | + |  |  |

- Case (1) follow from Case (3) by permuting the cylinder under gluing. For case (3), we mainly use axiom (3.1):

$$\begin{aligned}
\tau(C^{'})\circ\tau(C) &= \phi_{g_4^{-1}g_2^{-1}g_1g_2g_4}\Big(\varphi_{g_2^{-1}g_1g_2,g_4}\otimes 1\Big)\Big(\varphi_{g_1,g_2^{-1}}\otimes 1\Big)\\
&= \phi_{g_4^{-1}g_2^{-1}g_1g_2g_4}\Big(\varphi_{g_2^{-1}g_1g_2,g_4}\cdot\varphi_{g_1,g_2^{-1}}\otimes 1\Big)\\
&= \phi_{g_4^{-1}g_2^{-1}g_1g_2g_4}\Big(\varphi_{g_1,g_4^{-1}g_2^{-1}}\otimes 1\Big)\\
&= \tau(C_{--}(g_1,(g_2g_4)^{-1}))
\end{aligned}$$

- Case (2) We need to prove that:

$$\Big[\tau(C_{--}(g_1,g_2))\otimes 1\Big]\circ\Big[1\otimes\tau(C_{++}(g_3,g_4))\Big]=\tau\big(C_{-+}(g_1,g_2g_4)\big) \tag{3.22}$$

where $g_3=g_2^{-1}g_1^{-1}g_2$. Observe the obvious $\mathcal{X}$-homeomorphism between the following composition(gluing) of $(2,\mathcal{X})$-annuli:

$$\Big[C_{-+}(g_1,g_2^{-1})\Big]\circ\Big[C_{-+}(g_2^{-1}g_1g_2,1)\cup C_{++}(g_1,g_2)\Big]\cong_{\mathcal{X}}$$

$$\Big[C_{-+}(g_1,1)\cup C_{++}(g_1,g_2)\Big]\circ\Big[C_{-+}(g_1,g_2^{-1})\cup C_{-+}(g_1,1)\cup C_{-+}(g_2^{-1}g_1^{-1}g_2,1)\Big].$$

which algebraically can be interpreted as:

$$\big(\varphi_{g_1,g_2^{-1}}\otimes 1\otimes 1\big)\big(1\otimes F_{g_3,g_4}\big)=\big(1\otimes F_{g_3,g_4}\big)\varphi_{g_1,g_2^{-1}}$$

Now putting the values(morphisms) for the left side of equation (3.22), we get :

$$\begin{aligned}
\big(\phi_{g_2^{-1}g_1g_2}\otimes 1\big)\big(\varphi_{g_1,g_2^{-1}}\otimes 1\otimes 1\big)\big(1\otimes F_{g_3,g_4}\big) &= \big(\phi_{g_2^{-1}g_1g_2}\otimes 1\big)\big(1\otimes F_{g_3,g_4}\big)\varphi_{g_1,g_2^{-1}}\\
&= \varphi_{g_3^{-1},g_4^{-1}}\varphi_{g_1,g_2^{-1}}\\
&= \varphi_{g_1,(g_2g_4)^{-1}}\\
&= \tau\big(C_{-+}(g_1,g_2g_4)\big).
\end{aligned}$$

- Case (3) and case (4) follows from a similar type of argument. We have already discussed case (3) before. Case (4) is elaborated below:

$$\begin{aligned}\tau(C') \circ \tau(C) &= \varphi_{g_1, g_2^{-1}} \circ \varphi_{g_2^{-1} g_1 g_2, g_4} \\ &= \varphi_{g_1, g_4^{-1} g_2^{-1}} \\ &= \varphi_{g_1, (g_2 g_4)^{-1})} \\ &= \tau(C_{-+}(g_1, (g_2 g_4)^{-1}))\end{aligned}$$

- Case (5) follows from case (4) by simply permuting the cylinder under gluing.

- Case (6) We want to show

$$F_{g_3, g_4}\big(\varphi_{g_3, g_2} \otimes 1\big) = F_{g_1^{-1}, (g_2 g_4)}$$

Note that the $(2, \mathcal{X})$-annulus $C_{+-}(g_1, g_2)$ is obtained from a $(2, \mathcal{X})$-annulus of type $C_{-+}$ by reversing the boundaries and then changing their orientations. So $C_{+-}(g_1, g_2)$ can be obtained from $C_{-+}(g_2^{-1} g_1^{-1} g_2, g_2^{-1})$ as follows :

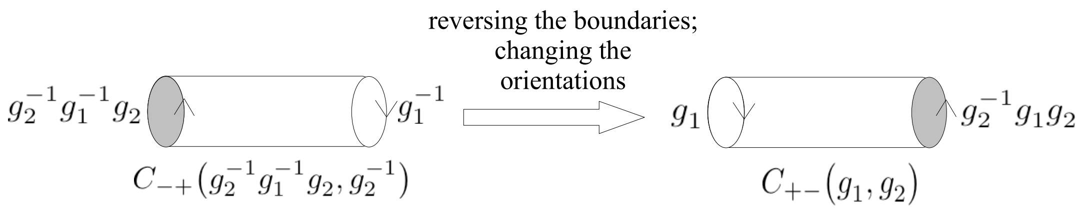



Figure 3.18: $C_{+-}(g_1, g_2)$

Thus, $C_{+-}(g_2^{-1} g_1^{-1} g_2, g_2^{-1})$ can be algebraically annotated by the following morphism:

$$\tau\Big(C_{+-}(g_2^{-1} g_1^{-1} g_2, g_2^{-1})\Big) = \varphi_{g_2^{-1} g_1^{-1} g_2,\, g_2}.$$

Observe that using gluing axiom, we have:

$$(\phi_{g_1} \otimes 1)(1 \otimes F_{g_3, g_4}) = (1 \otimes F_{g_3, g_4})(\phi_{g_1} \otimes 1).$$

$$\Rightarrow (\phi_{g_1} \otimes 1)(1 \otimes F_{g_3,g_4})(1 \otimes \varphi_{g_3,g_2} \otimes 1)$$

$$\begin{aligned}
&= (1 \otimes F_{g_3,g_4})(\phi_{g_1} \otimes 1)(1 \otimes \varphi_{g_3,g_2} \otimes 1)\\
&= (1 \otimes F_{g_3,g_4})\Big[\phi_{g_1}(1 \otimes \varphi_{g_3,g_2}) \otimes 1\Big]\\
&= (1 \otimes F_{g_3,g_4})\Big[\phi_{g_1}\big(\varphi_{g_2^{-1}g_1g_2,g_2}\varphi_{g_1,g_2^{-1}} \otimes \varphi_{g_3,g_2}\big) \otimes 1\Big] \quad \text{(By axiom 3.1)}\\
&= (1 \otimes F_{g_3,g_4})\Big[\phi_{g_2^{-1}g_1g_2}(\varphi_{g_1,g_2^{-1}} \otimes 1) \otimes 1\Big] \quad \text{( By axiom 3.3)}\\
&= (1 \otimes F_{g_3,g_4})\Big[\phi_{g_3^{-1}}(\varphi_{g_1,g_2^{-1}} \otimes 1) \otimes 1\Big] \quad \text{( as, } g_3 = g_2^{-1}g_1g_2\text{.)}\\
&= (1 \otimes F_{g_3,g_4})\big(\phi_{g_3^{-1}} \otimes 1\big)\big(\varphi_{g_1,g_2^{-1}} \otimes 1 \otimes 1\big)\\
&= \varphi_{g_3^{-1},g_4^{-1}}\varphi_{g_1,g_2^{-1}} \quad \text{( By 3.8)}\\
&= \varphi_{g_1,(g_2g_4)^{-1}} \quad \text{( By axiom 3.1)}\\
&= (\phi_{g_1} \otimes 1)\big(1 \otimes F_{g_1^{-1},(g_2g_4)^{-1}}\big) \quad \big(\text{By (3.17)}\big)
\end{aligned}$$

Comparing both sides, we get the required equality: $F_{g_3,g_4}\big(\varphi_{g_3,g_2} \otimes 1\big) = F_{g_1^{-1},(g_2g_4)}$.

- Case (7). This case follows from case(2) by permuting the $(2, \mathcal{X})$-annuli under gluing.

- Case (8). We glue $C$ with $C'$ along the common boundary $g_3$ and then use use axiom (3.1) to prove the gluing axiom of HQFT. We have

$$\big(1 \otimes \tau(C')\big) \circ \tau(C) = (1 \otimes \varphi_{g_3,g_4^{-1}})(1 \otimes \varphi_{g_1^{-1},g_2^{-1}})\Delta_{g_1,g_1^{-1}}\eta.$$

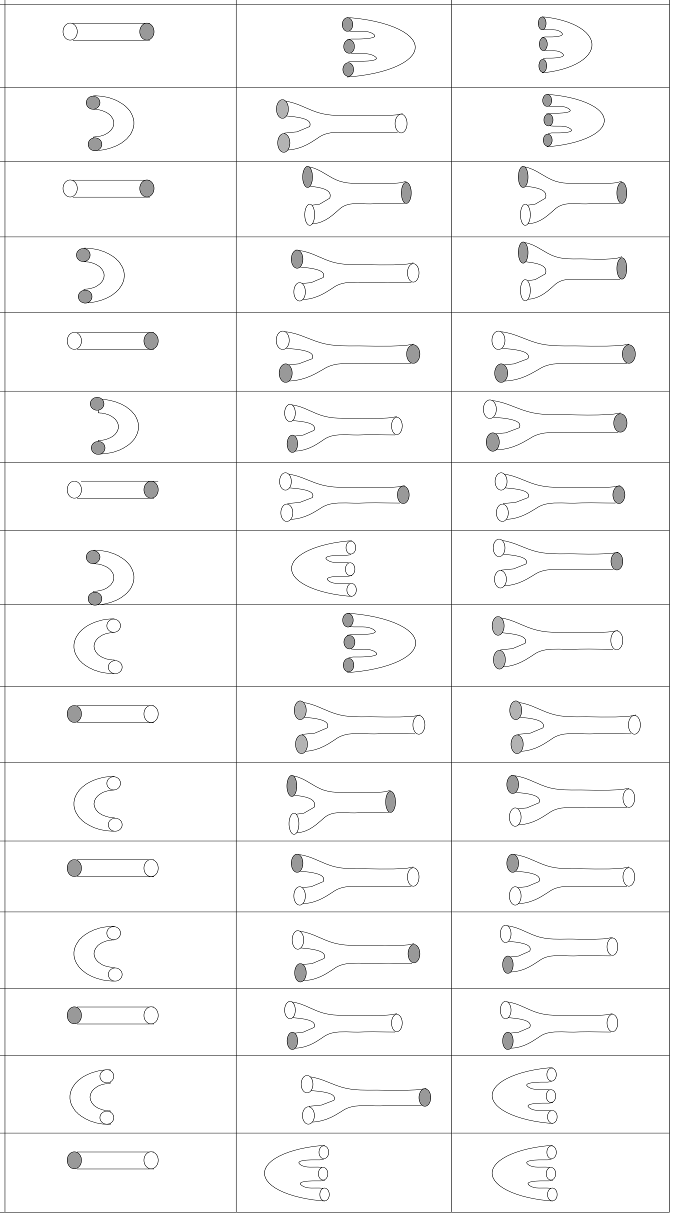
For this consider the diagram below:

$$\begin{tikzcd}
I \arrow[r, "\eta"] & A_1 \arrow[r, "\Delta_{g_1,g_1^{-1}}"] & A_{g_1} \otimes A_{g_1^{-1}} \arrow[r, "1\otimes\varphi_{g_1^{-1},g_2^{-1}}"] \arrow[dr, "1\otimes\varphi_{g_1^{-1},g_4^{-1}g_2^{-1}}"'] & A_{g_1} \otimes A_{g_2^{-1}g_1^{-1}g_2} \arrow[d, "1\otimes\varphi_{g_2^{-1}g_1^{-1}g_2,g_4^{-1}}"] \\
& & & A_{g_1} \otimes A_{g_4^{-1}g_2^{-1}g_1^{-1}g_2g_4}
\end{tikzcd}$$

The little triangle above commutes because of axiom (3.1).

Next we check the gluing condition when a $(2,\mathcal{X})$-annuli $C_{\epsilon_0,\epsilon_1}$ is composed with a $(2,\mathcal{X})$-pant $P_{\epsilon,\mu,\nu}$. By the topological invariance of $\tau$ , it is enough to consider the gluing performed along an $\mathcal{X}$-homeomorphism: $C^0_{\epsilon_0} = N_\nu$ so that $\epsilon_0 = -\nu$. We have 16 cases corresponding to different signs $\epsilon_1, \epsilon, \mu, \nu$. The table below shows all the cases; the shaded boundaries are with the negative orientation which shall indicate the inboundary of the pants/cylinders.

| | $\varepsilon_1$ | $\varepsilon$ | $\mu$ | $\nu$ | $C = C_{-\nu,\varepsilon_1}(g_1,g_2)$ | $P = P_{\varepsilon,\mu,\nu}(g_1,g_2,g_3,g_4)$ | Gluing C with P |
|---|---|---|---|---|---|---|---|
| 1 | - | - | - | - | | | |
| 2 | - | - | - | + | | | |
| 3 | - | - | + | - | | | |
| 4 | - | - | + | + | | | |
| 5 | - | + | - | - | | | |
| 6 | - | + | - | + | | | |
| 7 | - | + | + | - | | | |
| 8 | - | + | + | + | | | |
| 9 | + | - | - | - | | | |
| 10 | + | - | - | + | | | |
| 11 | + | - | + | - | | | |
| 12 | + | - | + | + | | | |
| 13 | + | + | - | - | | | |
| 14 | + | + | - | + | | | |
| 15 | + | + | + | - | | | |
| 16 | + | + | + | + | | | |



Consider gluing of any $(2,\mathcal{X})$-annulus of type $C_{-+}$ to any $(2,\mathcal{X})$-pants of type $P_{--+}$: if the gluing is performed along an $\mathcal{X}$-homeomorphism $C^1_+ \cong_{\mathcal{X}} L_-$ or $C^1_+ \cong_{\mathcal{X}} M_-$ then this follows from the identity $\varphi_{-,g}\varphi_{-,h} = \varphi_{-,gh}$ (Axiom 3.1); if the gluing is performed along $C^0_+ \cong_{\mathcal{X}} N_+$ then this follows from Axiom (3.2) of a Turaev crossed system.

Next, consider gluing of any $(2,\mathcal{X})$-annulus of type $C_{-+}$ to a $(2,\mathcal{X})$-pants of type

$P_{---}$. The gluing axiom holds for such a composition(such a gluing produces again $P_{---}$). If the gluing is performed along $L$ or $M$ then this follows from the identity . The existence of a self-$\mathcal{X}$-homeomorphism of $P_{---}$ mapping $N$ onto $L$ shows that the claim holds also for the gluing along $N$.

The cases where $\epsilon^0 = -\nu$ and the triple $\epsilon, \mu, \nu$ contains at least two minuses follows the arguments discussed above. The cases where $\epsilon = \mu$ are checked one by one using directly the definitions and the properties of $\tau$ established above, specifically, the gluing axiom for the $(2, \mathcal{X})$-annuli discussed above. The key argument in all these cases is that the tensor contractions along different tensor factors commute. The case $\epsilon = -$, $\mu = +$ reduces to $\epsilon = +$, and $\mu = -$ by the topological invariance. Assume that $\epsilon = +$, $\mu = -$. If $\nu = +$, $\epsilon^1 = +$ then the gluing axiom follows again from definitions. The remaining three cases ($\mu = +$, $\epsilon^1 = -$), and ($\nu = -$, $\epsilon^1 = \pm$) can be deduced using the multiplicativity of $\tau$, which we have discussed just before the lemma.

The gluing axiom for an HQFT holds for a composition of any cylinder of type $C_{-+}$ to $P_{--+}$: if the gluing is performed along an $\mathcal{X}$-homeomorphism $\partial_1 C_+ = L_-$ or $\partial_1 C_+ = M_-$ then this follows from the axiom(3.1); if the gluing is performed along $\partial_0 C_- = N_+$ then this follows from axiom (3.2). A cylinder of type $C_{-+}$ when glued to $P_{---}$ produces again $P_{---}$. If the gluing is performed along $L$ or $M$ then this follows from the axiom (3.1). The existence of a self-homeomorphism of $P_{---}$ mapping $N$ onto $M$ shows that the claim holds also for the gluing along $N$.

Now, consider the gluing of an $\mathcal{X}$-disc $D_+$ to an $(2, \mathcal{X})$-annulus $C_{-+}$. the gluing axiom follows from the equality $\varphi_{1,g} \circ \eta = \eta$ for all $g \in G$. Gluing axiom for a composition of $D_+$ to an $(2, \mathcal{X})$-pants $P_{---}$ follows from the equalities $\mu_{1,g}(\eta \otimes 1_g) = \mu_{g,1}(1_g \otimes \eta)$ for any $g \in G$. This and the definition of $\tau$ for $(2, \mathcal{X})$-pants of types $P_{---}$ or $P_{-++}$ imply the axiom for any gluing of $D_+$ to such pants. The gluing axiom for a composition of $D_-$ to $\mathcal{X}$-surfaces with$\leq$ 2 holes follows from the already established properties of the gluing of $C_{--}(1, 1)$ and $D_+$.

Now consider any connected $\mathcal{X}$-surface $W$. To compute $\tau(W)$ we need to compute it on a splitting system of loops. Turaev [Tur99] has argued that $\tau(W)$ neither depends on the choice of orientations and base points on $\alpha_1, \cdots, \alpha_N$ nor on the choice of a splitting system of loops on $W$. His crucial argument is provided by the fact (see [HT80]) that any two splitting systems of loops on $W$ are related by the following transformations: (i) isotopy in $W$ ; (ii) adding to a splitting system of loops $\alpha_1, \cdots, \alpha_N$ a simple loop $\alpha \subset W \cup_i \alpha_i$; (iii) deleting a loop from a splitting system of loops, provided the remaining loops form a splitting system; (iv) replacing one of the loops $\alpha_i$ of a splitting system adjacent to two different pants $P_1, P_2$ by a simple loop lying in $\mathrm{Int}P_1 \cup \mathrm{Int}P_1 \cup \alpha_i$, meeting $\alpha_i$ transversally in two points and splitting both $P_1$ and $P_2$ into annuli; (v) replacing one of the loops of a splitting system by a simple loop meeting it transversally in one point and disjoint from the other loops. He has checked the invariance of $\tau(W)$ under these transformations. □

**Theorem 3.5.3** *A Turaev $G$-crossed system $(A_g, \mu, \Delta, \epsilon)$ in $S$ defines (up to isomorphism) a (1+1)-dimensional symmetric HQFT over $K(G,1)$ space with values in $S$.*

Proof: We have realised Turaev $G$-crossed system $(A_g, \mu, \Delta, \epsilon)$ as the underlying system of (1+1)-dimensional $\mathcal{X}$-HQFT $(Z, \tau)$ with $X = K(G;1)$ using the Lemma (3.5.1) and Lemma (3.5.2).

□

Suppose $\mathcal{X}$ is an Eilenberg-MacLane space with $X = K(G;1)$ for a group $G$. Let us recall from Section (3.4.2) the category $\mathcal{T}(G,S)$ of Turaev crossed $G$-systems in $S$ and from Section (3.3) the category $\mathcal{Z}_2(\mathcal{X}, S)$ of (1+1)-dimensional $\mathcal{X}$-HQFTs with values in $S$. We wish to establish a functor from the category $\mathcal{Z}_2(\mathcal{X})$ to $\mathcal{T}(G,S)$. The functor is clear on objects. It simply takes any $\mathcal{X}$-HQFT to its underlying Turaev $G$-crossed system $A = \{A_g\}$ [Theorem (3.4.6)]. Let $\rho = \{\rho_{\mathcal{M}} : A_{\mathcal{M}} \to A^{'}_{\mathcal{M}}\}$ be a morphism between $\mathcal{X}$-HQFTs $(Z, \tau)$ and $(Z^{'}, \tau^{'})$. Suppose $A = \{A_g\}$ and $A^{'} = \{A^{'}_h\}$ are their respective underlying Turaev $G$-crossed systems in $S$. The morphism $\rho$ in $Z_2(\mathcal{X})$ corresponds to a morphism between $A$ and $A^{'}$ in $\mathcal{T}(G,S)$. Note that $\rho$ yields a collection $\{\rho_f : A_f \to A^{'}_f\}$ of morphisms in $S$ for the connected 1-dimensional $\mathcal{X}$-manifolds. The commutativity of the natural square diagrams associated with the $\mathcal{X}$-cobordisms $P_{--+}(f,g,1,1)$, $D_+$, $C_{--}(f,1)$, and $C_{-+}(f,g^{-1})$ together with the fact that $\rho_{\emptyset} = \mathrm{id}_{I_{\mathcal{C}}}$ in $Z_2(\mathcal{X})$ imply that the collection $\{\rho_f : A_f \to A^{'}_f\}$ of morphisms in $S$ commutes with the multiplication, and the action of $G$; and preserves the unit and the inner product(form). This establishes a functor

$$\mathcal{F} : \mathcal{Z}_2(\mathcal{X}, S) \to \mathcal{T}(G, S).$$

**Proposition 3.5.4** *Given a monoidal category $S$, if $\mathcal{X} = (X, x)$ is a $K(G;1)$ space, then the functor $\mathcal{F} : \mathcal{Z}_2(\mathcal{X}, S) \to \mathcal{T}(G, S)$ is an equivalence of categories.*

PROOF: The surjectivity for objects in $\mathcal{T}(G, S)$ has been established in Theorem (3.5.3). Let us now establish that for any two (1+1)-dimensional $\mathcal{X}$-HQFTs $(Z, \tau)$ and $(Z^{'}, \tau^{'})$ with values in $S$ and with the underlying Turaev $G$-crossed systems $A$, $A^{'}$, the homomorphism

$$\mathrm{Hom}\Big((Z,\tau),(Z^{'},\tau^{'})\Big) \to \mathrm{Hom}\,(A, A^{'}) \tag{3.23}$$

is bijective. The injectivity of the morphism in (3.23) is obvious since all 1-dimensional $\mathcal{X}$-manifolds are disjoint unions of loops and therefore any two morphisms $(Z, \tau) \to (Z^{'}, \tau^{'})$ coinciding on loops coincide on all 1-dimensional $\mathcal{X}$-manifolds.

Now to establish the surjectivity of the morphism in (3.23) note that every morphism of a $G$-crossed system $\rho : A \to A^{'}$ defines a morphism $Z_{\mathcal{M}} \to Z^{'}_{\mathcal{M}}$ for any connected 1-dimensional $\mathcal{X}$-manifold $\mathcal{M}$. These morphisms extend to non-connected $\mathcal{X}$-manifolds $\mathcal{M}$ by multiplicativity. We need to show that the resulting family of morphisms $\{\rho_M : Z_{\mathcal{M}} \to A^{'}\}_{\mathcal{M}}$ would make the natural square diagrams related to $\mathcal{X}$-homeomorphisms and $\mathcal{X}$-surfaces commutative. The part concerning the $\mathcal{X}$-homeomorphisms is obvious. As explained before, every $\mathcal{X}$-surface can be obtained by gluing from a finite collection of the basic $\mathcal{X}$-surfaces of type $D_+$, $C_{-+}(f, g)$, $C_{--}(f, 1)$, $C_{++}(f, g)$, and $P_{--+}(f, g, 1, 1)$. Therefore it suffices to check the commutativity of the square diagrams associated with these $\mathcal{X}$-surfaces. For $D_+$ and $P_{--+}(f, g, 1, 1)$ this follows from the assumption that $\rho$ commutes with multiplication and preserves the unit. For $C_{-+}(f, g)$, $C_{--}(f, 1)$, $C_{++}(f, g)$, this follows from the formula's (3.15) - (3.17) and the assumption that $\rho$ preserves the inner product and commutes with the action of $G$. Thus the morphism in (3.23) is bijective. This completes the proof. $\square$

## 3.6 Examples

The goal of this section is to construct examples of a Turaev crossed system which we defined in the Section 3.4. We start with a commutative ring $K$ with a unit. Let $\pi$ and $H$ be groups such that $\pi$ acts on $H$. Let $\theta$ be a normalised 2-cochain of $H$ with values in $K^{\times}$.

Here is a briefing of what we do in this section. In the first subsection we construct a Turaev crossed $\pi$-system in the category of $K$-modules starting from a crossed module $(H, \pi, t, \varphi)$. In the second subsection we define a twisted category $\mathcal{A}_{\pi}^{\sigma,\tau}$ where $(\sigma, \tau)$ is an abelian 3-cocycle for a group $\pi$ with values in $K^{\times}$ and discuss some properties of such categories in general. In the last subsection we work out another set of examples for a Turaev crossed $\pi$-system taking values in the twisted category $\mathcal{A}_{\pi}^{\sigma,\tau}$.

### 3.6.1 Crossed module

Given that $H$ and $\pi$ are multiplicative groups, let $(H, \pi, t, u)$ forms a crossed module where $t$ is a group homomorphism from $H$ to $\pi$ and $\pi$ acts on $H$ via $u$. Let $\theta = \{\theta(f, g) \in K^{\times}\}_{f,g\in H}$ be a normalised 2-cocycle of $H$. For any $\alpha \in T := \operatorname{Im} t$, define $L_{\alpha} = \bigoplus_{t(h)=\alpha} Kh$. Then each $L_{\alpha}$ is a $K$-module. Let us denote the collection of these $K$-modules as

$$\mathcal{L} := \{L_{\alpha} | \alpha \in T\}.$$

We aim to provide $\mathcal{L}$ with a structure of a Turaev $T$-crossed system. For $h, g$ in $H$ such that $t(h) = \alpha$ and $t(g) = \beta$ for some $\alpha, \beta \in T$, consider the following equations:

$$h \otimes g \mapsto \theta(h, g)hg,$$

$$k \mapsto k.1_H.$$

Extending these maps equips $\mathcal{L}$ with multiplications $\mu_{\alpha,\beta} : L_{\alpha} \otimes L_{\beta} \to L_{\alpha\beta}$ and a unit $\eta : K \to L_1$ which are all $K$-module morphisms. Let us define a map on $H \times H$ taking values in $K^{\times}$ as:

$$(f, g) \mapsto \theta(f, g) \tag{3.24}$$

if $fg = 1$ and zero otherwise. For any pair $f, g \in H$ such that $fg = 1$, $t(f) = \alpha$ and $t(g) = \beta$, for some $\alpha, \beta \in T$, the map given by (3.24) can be extended to a $K$-module morphism $\phi : L_\alpha \otimes L_\beta \to K$ which gives a pairing on $\mathcal{L}$.

Let us define a map on $H \times H$ taking values in $K^\times$ given as $< f, g >= \theta(f, g)\theta(fgf^{-1}, f)^{-1}$. We use this map to define a crossing on $\mathcal{L}$. Let us call $\mathrm{Mor}(\mathcal{L})$ to be the set of all morphisms of $K$-modules between different components of $\mathcal{L}$. Explicitly, it is given as:

$$\mathrm{Mor}(\mathcal{L}) = \bigcup_{\alpha \in T} \Big( \sqcap_{\beta \in T} \mathrm{Hom}\,(L_\beta, L_{\alpha\beta\alpha^{-1}}) \Big).$$

Then any element in $\mathrm{Mor}(\mathcal{L})$ is a collection of morphisms between different components of $\mathcal{L}$.

Let us define a map $\widetilde{\varphi} : H \to \mathrm{Mor}(\mathcal{L})$ by extending the map given below. For $f \in H$, such that $t(f) = \alpha$, the $\beta$-component of $\widetilde{\varphi}_f$, $\big(\widetilde{\varphi}_f\big)_\beta := \widetilde{\varphi}(f)_\beta : L_\beta \to L_{\alpha\beta\alpha^{-1}}$ is defined by extending

$$\widetilde{\varphi}_f(g) =< f, g > fgf^{-1} \tag{3.25}$$

for $g \in H$ such that $\beta = t(g)$. For any $f \in H$, we say $\widetilde{\varphi}_f = I$ if $\widetilde{\varphi}_f$ is identity on every component, that is, $\widetilde{\varphi}_f : L_g \to L_{fgf^{-1}}$ is identity for all $f \in G$ provided $gfg^{-1} = f$.

Let us now define a crossing $\varphi : T \to \mathrm{Mor}(\mathcal{L})$ for the system $\mathcal{L}$ as the extension of $\widetilde{\varphi}$ onto $T$.

Consider the following diagram:

$$\begin{tikzcd} H \arrow[r, "\widetilde{\varphi}"] \arrow[d, "t"'] & \mathrm{Mor}(\mathcal{L}) \\ T \arrow[ur, dotted, "\varphi"'] & \end{tikzcd}$$

For $\varphi$ to be well defined it suffices to show that $\varphi t|_Z = \widetilde{\varphi}|_Z = \mathrm{Id}$ where $Z = \mathrm{Ker}\, t$. Thus we have the following proposition:

**Proposition 3.6.1** *Given $g \in H$, if for all $z \in \mathrm{Ker}\, t$*

$$\theta(g, z) = \theta(z, g), \tag{3.26}$$

*then the crossing $\varphi : T \to \mathrm{Mor}(\mathcal{L})$ becomes a well defined map.*

PROOF: Note that $Z$ is central in $H$. Suppose $\widetilde{\varphi}|_Z = \mathrm{Id}$, then equation (3.25) restricted to $Z$ gives us:

$$\begin{aligned}
\widetilde{\varphi}_f(g) &= \langle f,g\rangle fgf^{-1} \quad ; \ f\in Z \\
g &= \langle f,g\rangle fgf^{-1} \\
g &= \langle f,g\rangle g \qquad\quad ; \text{ since } Z \text{ is central in } H. \\
1 &= \theta(f,g)\theta(fgf^{-1},f)^{-1} \\
\theta(g,f) &= \theta(f,g).
\end{aligned}$$

□

Now with the set of maps $\left\{\mu_{\alpha,\beta},\eta,\phi,\varphi_\beta\right\}_{\alpha,\beta\in T}$, the system $\mathcal{L}$ becomes a Turaev $T$-crossed system. Details are worked out in the following result:

**Theorem 3.6.2** *Let $(H,\pi,t,u)$ be a crossed module. Let $\theta$ be a normalised 2-cocycle of $H$ with values in $K^\times$. If for all $z\in \operatorname{Ker} t$ and $h\in H$, $\theta(h,z)=\theta(z,h)$ then $\mathcal{L}$ is a Turaev $T$-crossed system in the category of $K$-modules.*

Proof: For any $f\in H$, $\phi(f,f^{-1})=\theta(f,f^{-1})\neq 0$. This implies $\phi$ is a non-degenerate pairing of $H$. Thus the pairing $\phi$ when extending to components of $\mathcal{L}$ will also be non-degenerate. The associativity of multiplication defined above follows from (2.7). Substituting $g=f^{-1}$ and $h=f$ in (2.7) we get that $\theta(f,f^{-1})=\theta(f^{-1},f)$ for all $f\in H$. Hence the form $\phi$ is symmetric. Multiplicativity of $\varphi$ follows from the direct computation using mainly equation (2.7). Consider the coefficients in the expression for $\varphi_h(\varphi_f(g))$:

$$\begin{aligned}
<h,fgf^{-1}><f,g> &= \underline{\theta(h,fgf^{-1})}\theta(hfgf^{-1}h^{-1},h)^{-1}\ \theta(f,g)\underline{\theta(fgf^{-1},f)}^{-1} \\
&= \underline{\theta(h,fg)}\theta(hfgf^{-1}h^{-1},f)^{-1}\ \theta(hfgf^{-1}h^{-1},h)^{-1}\underline{\theta(f,g)} \\
&= \underline{\theta(h,f)}\theta(hf,g)\theta(hfgf^{-1}h^{-1},f)^{-1}\ \underline{\theta(hfgf^{-1}h^{-1},h)}^{-1} \\
&= \underline{\theta(hfgf^{-1}h^{-1},f)}\theta(hf,g)\underline{\theta(hfgf^{-1}h^{-1},f)}^{-1}\ \underline{\theta(hfgf^{-1}h^{-1},h)}^{-1} \\
&= \underline{\theta(hfgf^{-1},f)}\theta(hf,g)\ \underline{\theta(hfgf^{-1},f)}^{-1}\theta(hfgf^{-1}h^{-1},hf)^{-1} \\
&= \theta(hf,g)\ \theta(hfgf^{-1}h^{-1},hf)^{-1} \\
&= <hf,g>
\end{aligned}$$

which is the coefficient in the expression for $\varphi_{hf}(g)$. In the above calculations the terms which are underlined are either clubbed together for the next step or they are inverses of each other and hence cancelled out in the next step.

Let us verify axiom (3.2) of the Definition (3.4.1). For $f, h \in H$ and $\beta \in T$ such that $\beta = t(g)$, consider the coefficients in the expression for $\varphi_g(f \otimes h) = <g, fh> \theta(f, h)\big(g(fh)g^{-1}\big)$, which are :

$$
\begin{aligned}
<g, fh>\ \theta(f, h) &= \underline{\theta(g, fh)}\theta(fh, g)^{-1}\underline{\theta(f, h)} \\
&= \theta(g, f)\theta(gf, h)\theta(fh, g)^{-1}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\varphi_g(f) \otimes \varphi_g(h) &= \Big[<g, f> gfg^{-1}\Big] \otimes \Big[<g, h> ghg^{-1}\ \Big] \\
&= <g, f><g, h>\big(gfg^{-1} \otimes ghg^{-1}\big) \\
&= <g, f><g, h>\ \theta(gfg^{-1}, ghg^{-1})\big(gfhg^{-1}\big) \\
&= <g, f><g, h>\ \theta(f, h)\big(gfhg^{-1}\big).
\end{aligned}
$$

So, comparing the coefficients, we have $<g, f><g, h>\ \theta(gfg^{-1}, ghg^{-1})$

$$
\begin{aligned}
&= \theta(g, f)\underline{\theta(gfg^{-1}, g)}^{-1}\ \underline{\theta(g, h)}\theta(ghg^{-1}, g)^{-1}\theta(gfg^{-1}, ghg^{-1}) \\
&= \theta(g, f)\theta(gf, h)\ \underline{\theta(gfg^{-1}, gh)}^{-1}\ \theta(ghg^{-1}, g)^{-1}\underline{\theta(gfg^{-1}, ghg^{-1})} \\
&= \theta(g, f)\theta(gf, h)\underline{\theta(gfg^{-1}, g)}\ \theta(gfhg^{-1}, g)^{-1}\ \underline{\theta(ghg^{-1}, g)}^{-1} \\
&= \theta(g, f)\ \theta(gf, h)\ \theta(gfhg^{-1}, g)^{-1}.
\end{aligned}
$$

This implies that $\varphi_g(f \otimes h) = \varphi_g(f) \otimes \varphi_g(h)$. For axiom (3.3) of the Definition (3.4.1), consider for $h, h'$ in $H$ such that $hh' = 1$, the scalars in the expression $\phi\big(\varphi_g(h) \otimes \varphi_g(h')\big)$

becomes :

$$
\begin{aligned}
&= \theta(ghg^{-1}, gh^{'}g^{-1}) <g,h> <g,h^{'}> \\
&= \underline{\theta(ghg^{-1}, gh^{'}g^{-1})}\theta(ghg^{-1}, g)^{-1}\theta(g,h)\underline{\theta(gh^{'}g^{-1}, g)}^{-1}\theta(g,h^{'}) \\
&= \theta(ghh^{'}g^{-1}, g)^{-1}\underline{\theta(ghg^{-1}, gh^{'})}\theta(ghg^{-1}, g)^{-1}\theta(g,h)\underline{\theta(g,h^{'})} \\
&= \underline{\theta(ghg^{-1}, g)}\theta(gh,h^{'})\underline{\theta(ghg^{-1}, g)}^{-1}\theta(g,h) \\
&= \theta(gh,h^{'})\theta(g,h) \\
&= \theta(h,h^{'})\theta(g,hh^{'}) \\
&= \theta(h,h^{'}).
\end{aligned}
$$

Note that $\theta(ghh^{'}g^{-1}, g)^{-1} = \theta(g, hh^{'}) = 1$ since $hh^{'} = 1$. The coefficient of the expression $\phi(h \otimes h^{'})$ is also $\theta(h,h^{'})$. This proves axiom (3.4). The way we have defined $\varphi_g$, axiom (3.5) follows directly. Let us check the last axiom for the trace. For a fixed element $c \in L_{t(m)}$ where $m = fhf^{-1}h^{-1}$, the linear map $b = c \otimes \varphi_h : L_m \otimes L_f \to L_f$ sends $l_f$ to $kl_f$ for some $l_f \in L_f$ and $k \in K$. The linear map $b^{'} = \varphi_{f^{-1}}(m \otimes 1) : L_m \otimes L_h \to L_h$ sends $l_h$ to $k^{'}l_h$ for some $l_h \in L_h$ and $k^{'} \in K$. Note that,

$$
k(f \otimes g) = (kf) \otimes g = \Big(c \otimes \varphi_g(f)\Big) \otimes g = c \otimes \Big(\varphi_g(f) \otimes g\Big) = c \otimes (g \otimes f) = (c \otimes g) \otimes f
$$

$$
= f \otimes \Big(\varphi_{f^{-1}}(c \otimes g)\Big) = f \otimes (k^{'}g) = k^{'}(f \otimes g).
$$

Therefore $k = k^{'}$. Thus Trace$(b) = k = k^{'} =$ Trace$(b^{'})$. Hence $\mathcal{L}$ forms a Turaev $T$-crossed system in $K$-modules. □

### 3.6.2 Twisted category

Suppose $G, M$ are multiplicative abelian groups. Then $M$ can be regarded as a trivial $G$-module (via the action $x\mu = \mu$ for $x \in G$, $\mu \in M$ ). An *abelian* $3-$*cocycle* for $G$ with coefficients in $M$ is a pair $(\sigma, \tau)$ , where $\sigma : G^3 \to M$ is a normalised 3-cocycle

$$
\sigma(a,1,b) = 1
$$

$$
\sigma(a,b,c)\sigma(d,ab,c)\sigma(d,a,b) = \sigma(d,a,bc)\sigma(da,b,c) \tag{3.27}
$$

and $\tau : G^2 \to M$ is a function satisfying :

$$\sigma(b,c,a)\tau(a,bc)\sigma(a,b,c) = \tau(a,c)\sigma(b,a,c)\tau(a,b) \tag{3.28}$$

$$\sigma(c,a,b)^{-1}\tau(ab,c)\sigma(a,b,c)^{-1} = \tau(a,c)\sigma(a,c,b)^{-1}\tau(b,c). \tag{3.29}$$

where $a,b,c \in G$.

A 2-cochain is a function $\theta : G^2 \to M$ satisfying :

$$\theta(a,1) = \theta(1,b) = 1.$$

The coboundary of $\theta$ is the abelian 3-cocycle $\partial(\theta) = (\sigma,\tau)$ defined by the equations

$$\sigma(a,b,c) = \theta(b,c)\theta(ab,c)^{-1}\theta(a,bc)\theta(a,b)^{-1} \tag{3.30}$$

$$\tau(a,b) = \theta(a,b)\theta(b,a)^{-1}. \tag{3.31}$$

Then $H^3_{\mathrm{ab}}(G,M)$ is the abelian group of 3-cocycles modulo the coboundaries. It might look difficult to compute these groups but Theorem 3.6.4 comes for our rescue. After proving the theorem, we will, in particular, compute these groups when $G$ is a cyclic group.

Suppose $K$ is a commutative ring with unit. Suppose $(\sigma,\tau)$ is an abelian 3-cocycle on $G$ with coefficients in the multiplicative group $K^\times$. We construct a twisted category $\mathcal{A}_G^{\sigma;\tau}$( or simply $\mathcal{A}$) of $G$-graded $K$-modules. The objects of $\mathcal{A}$ are

$$L = \bigoplus_{\alpha \in G} L_\alpha$$

where $\{L_\alpha\}_{\alpha\in G}$ is a family of $K$-modules. The arrows are direct sums of $K$-module homomorphisms. The tensor on this category is given by the following formula:

$$(L \otimes M)_\alpha = \sum_{\beta.\gamma=\alpha} L_\beta \otimes M_\gamma.$$

where $\alpha,\beta,\gamma \in G$. The following equations define an associativity constraint $a$ and a braiding $c$ in $\mathcal{A}$

$$a((x \otimes y) \otimes z) = \sigma(\alpha, \beta, \gamma)(x \otimes (y \otimes z))$$

$$c(x \otimes y) = \tau(\alpha, \beta) y \otimes x$$

where $x \in L_\alpha, y \in M_\beta, z \in N_\gamma$. We then extend the definition by linearity to define it on the objects of the category. In this way, $\mathcal{A}$ becomes a monoidal category of $G$-graded $K$-modules. In other words, $\mathcal{A}$ is a monoidal category of $KG$-comodules with associativity given by $\sigma$.

**Proposition 3.6.3** *Let $(\sigma, \tau)$ and $(\sigma^{'}, \tau^{'})$ be any two representatives of their equivalence classes. If $[\sigma, \tau] = [\sigma^{'}, \tau^{'}] \in H^3_{ab}(G, K^\times)$, then we get a braided tensor isomorphism between the categories $\mathcal{A}^{\sigma,\tau}_G$ and $\mathcal{A}^{\sigma^{'},\tau^{'}}_G$.*

Proof: Consider the identity functor $I$ from the category $\mathcal{A}^{\sigma,\tau}_G$ to the category $\mathcal{A}^{\sigma^{'},\tau^{'}}_G$. We claim that it is in fact a braided tensor isomorphism. Let $k : G^2 \to M$ be a function satisfying $k(a, 1) = k(1, b) = 1$, such that

$$(\sigma, \tau)(\sigma^{'}, \tau')^{-1} = \partial(k).$$

Define tensor isomorphism via the natural transformation $\phi_{L,M} : I(L) \otimes I(M) \to I(L \otimes M)$ as $\phi_{L,M}(\sum l_\beta \otimes m_\gamma) = \sum k(\beta, \gamma)(l_\beta \otimes m_\gamma)$, where $l_\beta \in L_\beta$ and $m_\gamma \in M_\gamma$ for $\alpha, \beta \in \pi$. Let us first show that $\phi$ is a tensor functor. Given any three objects $M, N, L$ in the category $\mathcal{A}^{\sigma,\tau}_G$, we want the the following condition is satisfied

$$a \cdot (\phi_{M \otimes N, L}) \cdot (\phi_{M,N} \otimes 1_L) \quad = \quad \phi_{M,(N \otimes L)} \cdot (1_M \otimes \phi_{N,L}) \cdot a^{'}.$$

For any $b, c, e \in G$ , when a typical element $\sum(m_b \otimes n_c) \otimes l_e$ is applied to the equation above, the left hand side becomes :

$$
\begin{aligned}
&= a \cdot (\phi_{M\otimes N,L}) \cdot (\phi_{M,N} \otimes 1_L)(\sum (m_b \otimes n_c) \otimes l_e) \\
&= a \cdot (\phi_{M\otimes N,L})(\sum \phi_{M,N}(m_b \otimes n_c) \otimes l_e)) \\
&= a \cdot (\phi_{M\otimes N,L})(\sum k(b,c)(m_b \otimes n_c) \otimes l_e) \\
&= a(\sum k(bc,e)(\sum k(b,c)(m_b \otimes n_c) \otimes l_e) \\
&= \sum \sigma(b,c,e)\, k(bc,e)\, k(b,c)(m_b \otimes (n_c \otimes l_e))
\end{aligned}
$$

and after applying $\sum(m_b \otimes n_c) \otimes l_e$ to the equation above, the right hand side of the equation becomes :

$$
\begin{aligned}
&= \phi_{M,N\otimes L} \cdot (1_M \otimes \phi_{N,L}) \cdot a^{'}(\sum (m_b \otimes n_c) \otimes l_e) \\
&= \phi_{M,N\otimes L} \cdot (1_M \otimes \phi_{N,L})(\sum \sigma^{'}(b,c,e) m_b \otimes (n_c \otimes l_e)) \\
&= \phi_{M,N\otimes L}(\sum \sigma^{'}(b,c,e) m_b \otimes \phi_{N,L}(n_c \otimes l_e)) \\
&= \phi_{M,N\otimes L}(\sum \sigma^{'}(b,c,e)\, k(c,e) m_b \otimes (n_c \otimes l_e)) \\
&= \sum \sigma^{'}(b,c,e)\, k(b,ce)\, k(c,e)(m_b \otimes (n_c \otimes l_e))
\end{aligned}
$$

The coefficients of the left side $\sigma(b,c,e)\, k(bc,e)\, k(b,c)$ equals the coefficients on the right side $\sigma^{'}(b,c,e)\, k(b,ce)\, k(c,e)$ using (3.15) and $(\sigma,\tau)(\sigma^{'},\tau')^{-1} = \partial(k)$. To show that the isomorphism is also braided we need to show the following

$$
\phi_{N,M} \cdot c^{'}_{M,N} \quad = \quad c_{M,N} \cdot \phi_{M,N}
$$

For any $b, c \in G$ and using the equation $\tau(b,c) = k(b,c)k^{-1}(c,b)$ one can show that the isomorphism is also braided. Applying a general element $\sum m_b \otimes n_c$, the left hand side of the equation above becomes :

$$
\begin{aligned}
&= \phi_{M,N} \cdot c^{'}_{M,N}(\sum m_b \otimes n_c) \\
&= \phi_{M,N}(\sum \tau^{'}(b,c) n_c \otimes m_b) \\
&= \sum \tau^{'}(b,c) k(c,b) n_c \otimes m_b
\end{aligned}
$$

Applying $\sum m_b \otimes n_c$, the right hand side of the equation above becomes :

$$\begin{aligned} &= c_{M,N} \cdot \phi_{M,N}(\sum m_b \otimes n_c) \\ &= c_{M,N}(\sum k(b,c) m_b \otimes n_c \\ &= \sum \tau(b,c) k(b,c) n_c \otimes m_b \end{aligned}$$

The coefficient of left side $\tau^{'}(b,c)\, k(c,b)$ equals the coefficient on right side $\tau(b,c)\, k(b,c)$ using (3.16) and $(\sigma,\tau)(\sigma^{'},\tau')^{-1} = \partial(k)$. This completes the proof.

□

We recall the definition of a quadratic function. A function $q : G \to M$ between two abelian groups $G, M$ is called quadratic when $q(xy)q(x)^{-1}q(y)^{-1}$ is a bilinear function of $x, y$. This amounts to the following two conditions:

(i) $q(xyz)\ q(xy)^{-1}q(yz)^{-1}q(zx)^{-1}q(x)\ q(y)\ q(z) = 1$,

(ii) $q(x) = q(x^{-1})$.

Now to any abelian 3-cocycle $(\sigma, \tau)$ we assign the function $q(x) = \tau(x,x) \in M$ as its *trace*. By (3.21), the trace of a coboundary is identity, and one can show that the trace satisfies the above two identities. Thus traces are quadratic functions. We have the following result which appears in several papers without a proof. We discuss its proof here:

**Theorem 3.6.4** *([Mac52], [Eil52], [EM53]) The function assigning to each abelian 3-cocycle its trace induces an isomorphism of $H^3_{ab}(G,M)$ to the group of all quadratic functions on $G$ to $M$.*

Proof: For a fixed $M$, trace is natural in abelian groups $G$. Note that every abelian group is a filtered colimit of finitely generated abelian groups. Moreover, every finitely generated abelian group is a finite direct sum of cyclic groups. So it suffices to verify the isomorphism when $G$ is cyclic as quadratic functions into $M$ preserve filtered colimits.

Surjectivity: Suppose $q : G \to M$ is a quadratic function. Put $\nu = q(1)$. Define $\tau(x,y) = \nu^{xy}$. If $G$ is infinite, $(1,\tau)$ is an abelian 3-cocycle whose trace is $q$. If $G$ has order $n$, notice that $\nu^{2n} = \nu^{n^2} = 1$ in order for $q$ to be well defined. Let $\sigma$ be the 3-cocycle defined with trace $q$. The standard form for such a $\sigma$ is (cf. Chapter-2, Section 2.3)

$$\sigma(x,y,z)=\begin{cases}1 & \text{for } y+z<n\\ \nu^{xn} & \text{for } y+z\geq n\end{cases} \tag{3.32}$$

Then $(\sigma,\tau)$ is an abelian 3-cocycle.

Injectivity: It is done in the following two cases.

**Case 1 :** $G=\mathbb{Z}$. Let $(\sigma,\tau)$ be an abelian 3-cocycle such that $\tau(x,x)=1$. Since $H^3(G,M)=1$, we have $\sigma=d\theta$ for some 2-cochain $\theta$. Let

$$(\sigma^{'},\tau^{'})=(\sigma,\tau)\partial\theta^{-1}=(1,\tau^{'}).$$

Then $\tau^{'}(x,y)=\tau(x,y)\theta(x,y)^{-1}\theta(y,x)$. Substituting $\sigma^{'}=1$ in (3.28) and (3.29), we get

$$\tau^{'}(x,yz)=\tau^{'}(x,z)\tau^{'}(x,y),$$

$$\tau^{'}(xy,z)=\tau^{'}(x,z)\tau^{'}(y,z).$$

Equating for $\tau^{'}(x,z)$ in the above two equations, we get

$$\tau^{'}(xy,z)\tau^{'}(y,z)=\tau^{'}(xy,z)\tau^{'}(x,y).$$

Substitute $x=1$, $z=y^{-1}$ in the above equation, we get

$$\tau^{'}(1,1)\tau^{'}(y,y^{-1})=\tau^{'}(y,y^{-1})\tau^{'}(1,y).$$

Thus, $\tau^{'}(1,y)=1$. Similarly we can show $\tau^{'}(y,1)=1$. Then using (3.28) and (3.29) and that $\sigma^{'}=1$, $\tau^{'}(x,y)=1$ follows by induction.

**Case 2 :** $G=\mathbb{Z}_n$. Let $(\sigma,\tau)$ be an abelian 3-cocycle such that $\tau(x,x)=1$. Choose a 2-cochain $\theta$ such that $(\sigma^{'},\tau^{'})=(\sigma,\tau)\partial\theta^{-1}$ and $\sigma^{'}$ is standard for some $\nu$, as in (3.32). Similar to the Case (1), we have $\tau^{'}(x,x)=1$. Substitute $b=1$ in (3.29). We get

$$\sigma^{'}(1,c,a)=\sigma^{'}(1,a,c)\tau^{'}(a,1).$$

Since $G$ is abelian (3.32) implies that $\sigma^{'}(1,c,a) = \sigma^{'}(1,a,c)$. Thus, from the above equation, $\tau^{'}(a,1) = 1$. Similarly we can show $\tau^{'}(1,a) = 1$. Then using again (3.28) and (3.29) and the standard form of $\sigma^{'}$, one deduces by induction that $\tau^{'}(x,y) = 1$. Then the standard form for $\sigma^{'}$ reduces to 1. This proves injectivity.

□

We are now in a better position to calculate $H^3_{ab}(G,M)$ using the above theorem. In the case when $G$ is an infinite cyclic group $\mathbb{Z}$, let $q : G \to M$ be a quadratic function. Then it is simply a group homomorphism. Indeed the group of all quadratic functions on $\mathbb{Z}$ to any abelian group $M$ is $M$ itself, thus $H^3_{ab}(\mathbb{Z}, M) = M$. In the case when $G = \mathbb{Z}_n$ is a cyclic group of order $n$, let $q : \mathbb{Z}_n \to M$ be a quadratic function on $G$. Then it is simply a group homomorphism such that $q(1)^{2n} = q(1)^{n^2} = 1$ for $q$ to be a well defined quadratic function. Thus the group of all quadratic functions on $\mathbb{Z}/n$ to $M$ is isomorphic to $\mathrm{Hom}\big(\mathbb{Z}/(n^2, 2n), M\big)$. Hence,

$$H^3_{ab}(\mathbb{Z}/n, M) \cong \mathrm{Hom}\big(\mathbb{Z}/(n^2, 2n), M\big).$$

Consider for example $M$ as $\mathbb{C}^{\times}$ and $n = 6$ so that,

$$H^3_{ab}(\mathbb{Z}/6, \mathbb{C}^{\times}) \cong \mathrm{Hom}\,(\mathbb{Z}/(n^2, 2n), \mathbb{C}^{\times}) = \mathrm{Hom}\,(\mathbb{Z}/12, \mathbb{C}^{\times}) \cong \mathbb{Z}/12.$$

An explicit formula for defining a 3-cocycle can be worked out using the proof of Theorem (3.6.4). Accordingly, we define a 3-cocycle (h,c) as follows. Let $\nu = e^{2\pi\iota/12} \in \mathbb{C}^{\times}$ then $\nu^{6^2} = \nu^{2\times 6} = 1$. Define,

$$h(x,y,z) = \begin{cases} 1 & \text{for } yz < n \\ \nu^{xn} & \text{for } yz \geq n \end{cases}$$

$$c(x,y) = \nu^{xy}.$$

where $x, y, z \in \mathbb{Z}/n$ and $\nu = c(1,1)$. One can easily check that $(h,c)$ so defined is an abelian 3-cocycle.

### 3.6.3 Crossed module in a twisted category

Let $\pi$ and $H$ be multiplicative groups such that $t$ is a group homomorphism from $H$ to $\pi$ and $\pi$ acts on $H$ via $u$. Let $(H, \pi, t, u)$ forms a crossed module and let $T = \mathrm{Im}\, t$. Let $^{-} : H \twoheadrightarrow G$ maps $H$ onto an abelian group $G$ such that the following diagram commutes,

$$\begin{tikzcd} & T \\ H \arrow[ur, two heads, "t"] \arrow[r, two heads, "-"'] & G \arrow[u, "\widetilde{t}"'] \end{tikzcd} \tag{3.33}$$

that is, $\widetilde{t}(g) = t(h)$ for $g = \bar{h}$. Then $\widetilde{t}$ is also a surjective homomorphism. Note that surjection of $^{-}$ is not required for $\widetilde{t}$ to be an epimorphism, but we require its surjectivity later on. Let $\sigma = \{\sigma(\alpha, \beta, \gamma) \in K^{\times}\}_{\alpha,\beta,\gamma\in\pi}$ be a normalised 3-cocycle of $\pi$. And let $\theta = \{\theta(f, g) \in K^{\times}\}_{f,g\in H}$ be a normalised 2-cochain of $H$ such that $d\theta = \sigma|_H = \sigma\circ(t\times t\times t)$. Thus, for any $f, g, h \in H$, if $t$ maps them respectively into $\alpha, \beta, \gamma$ in $\pi$ then we have

$$\theta(f, g)\theta(fg, h) = \sigma(\alpha, \beta, \gamma)\theta(f, gh)\theta(g, h). \tag{3.34}$$

Let $\overline{\theta} = \{\theta(\overline{f}, \overline{g}) \in K^{\times}\}_{\overline{f},\overline{g}\in G}$. Then surjectivity of $^{-}$ implies that $\overline{\theta}$ is a 2-cochain of $G$. Then the following equation

$$d\overline{\theta} = \sigma|_G = \sigma \circ (\widetilde{t} \times \widetilde{t} \times \widetilde{t}) \tag{3.35}$$

again produces the same equation as before. Here $d\overline{\theta}$ That is, for any $\overline{f}, \overline{g}, \overline{h} \in G$ with their respective pre-images $f, g, h \in H$ with $\widetilde{t}(\overline{f}) = \alpha$, $\widetilde{t}(\overline{g}) = \beta$ , $\widetilde{t}(\overline{h}) = \gamma$, equation (3.34) gives:

$$\overline{\theta}(\overline{f}, \overline{g})\overline{\theta}(\overline{f}\overline{g}, \overline{h}) = \sigma(\alpha, \beta, \gamma)\overline{\theta}(\overline{f}, \overline{g}\overline{h})\theta(\overline{g}, \overline{h}),$$

which gives back equation (3.22). Let $(\sigma, \tau)$ be an abelian 3-cocycle of $T$ with coefficients in the multiplicative group $K^{\times}$. By abuse of notation, we set $\sigma = \sigma\widetilde{t}$ and $\tau = \tau\widetilde{t}$ where

$\sigma$ and $\tau$ on the right hand side correspond to the pullbacks under $\widetilde{t}$. For $\alpha \in T$, define $L_\alpha = \bigoplus_{t(h)=\alpha} Kh$. Then each $L_\alpha$ is an $KH$-comodule. Note that commutativity of the above diagram turns each $L_\alpha$ into a $KG$-comodule where

$$\left(L_\alpha\right)_g = \sum_{t(h)=\alpha,\, \bar{h}=g} Kh = \sum_{\widetilde{t}(\bar{h})=\alpha} Kh.$$

Consider the category $\mathcal{A}_G^{\sigma,\tau}$ as defined in the last subsection and $\sigma$ and $\tau$ are the restrictions to $G$.

Let $\mathcal{L} = \{L_\alpha : \alpha \in T\}$ be the collection of $KG$-comodules. Our goal is to provide this system with a structure of Turaev $T$-crossed system in $\mathcal{A}_G^{\sigma,\tau}$,

The multiplications of $\mathcal{L}$ can be obtained by extending the map $G \times G \to KG$ given by :

$$(f, g) \mapsto \theta(f, g) fg. \tag{3.36}$$

Observe that since $G$ is abelian, we have :

$$L_{\alpha\beta} = L_{\widetilde{t}(g)\widetilde{t}(h)} = L_{\widetilde{t}(gh)} = L_{\widetilde{t}(hg)} = L_{\beta\alpha}, \tag{3.37}$$

for $\alpha, \beta \in T$ such that $\widetilde{t}(g) = \alpha$ and $\widetilde{t}(h) = \beta$.

Note that $L_1 = \bigoplus_{t(h)=1} Kh = \bigoplus_{h\in \mathrm{Ker}\,(t)} Kh$. Thus the unit element $e$ of $G$ is also in $L_1$ which gives the identity of $\mathcal{L}$.

We define a pairing $\phi$ on $\mathcal{L}$ in a similar way as done in section (3.6.1) which will be non-degenerate as well as symmetric as before.

We define the dual $L_\alpha^*$ of $L_\alpha$ as $L_{\alpha^{-1}}$. Then $L_\alpha^* = \bigoplus_{\alpha^{-1}=\widetilde{t}(h)} Kh$. Let $\mathcal{L}^* = \{L_\alpha^* : \alpha \in T\}$. The multiplications for the system $\mathcal{L}^*$ is given by the same rule (3.23) as for $\mathcal{L}$. The unit for $\mathcal{L}^*$ is also the same as for $\mathcal{L}$ since $L_1^* = L_1$. For each $\alpha \in T$, $L_\alpha \cong L_\alpha^*$ as $KG$-comodules. In fact they are equal to each other as $L_{\alpha^{-1}}^* = L_\alpha$. We have the following result :

**Theorem 3.6.5** *Let $(H, \pi, t, u)$ be a crossed module such that $^- : H \twoheadrightarrow G$ is a surjective homomorphism of $H$ onto abelian group $G$ with the diagram (3.23) being commutative. Let $T = \mathrm{Im}\, t$. Let $\theta$ be a normalised 2-cochain of $H$ and $(\sigma, \tau)$ be an abelian 3-cocycle of*

*$T$. If*

$$d\theta = \sigma|_H$$

*then the collection $\mathcal{L} = \{L_\alpha | \alpha \in T\}$ forms a Frobenius $T$-graded system in $\mathcal{A}_G^{\sigma;\tau}$.*

PROOF: The multiplication given by the rule (3.36) induces isomorphisms $\mu_{\alpha,\beta} : (L_\alpha) \otimes (L_\beta) \to L_{\alpha\beta}$ such that

$$\alpha * \beta = \theta(\alpha, \beta)\, \alpha\beta. \tag{3.38}$$

for $\alpha$, $\beta \in T$. These form the multiplications for $\mathcal{L}$. Associativity of these multiplications is shown below. Let $\alpha$, $\beta$ and $\gamma$ be in $T$ such that their pre images in $G$ are $f, g$ and $h$ respectively. Then consider,

$$\begin{aligned}
(\alpha * \beta) * \gamma &= \theta(f, g)((\alpha\beta) * \gamma) \\
&= \theta(f, g)\theta(fg, h)(\alpha\beta)\gamma \\
&= \sigma(\alpha, \beta, \gamma)\theta(g, h)\theta(f, gh)(\alpha\beta)\gamma \\
&= \sigma(\alpha, \beta, \gamma)\theta(g, h)\theta(f, gh)(\alpha(\beta\gamma)) \\
&= \sigma(\alpha, \beta, \gamma)\theta(g, h)(\alpha * (\beta\gamma)) \\
&= \sigma(\alpha, \beta, \gamma)(\alpha * (\beta * \gamma)).
\end{aligned}$$

Hence associativity of multiplications for $\mathcal{L}^*$ is also proved. Thus $\mathcal{L}$ is a rigid $T$-algebra in $\mathcal{A}_G^{\sigma;\tau}$. Infact the two systems $\mathcal{L}$ and $\mathcal{L}^*$ are isomorphic in $\mathcal{A}_G^{\sigma;\tau}$ as $\mathcal{L}$-modules. Thus using Theorem 3.2.8, $\mathcal{L}$ is a Frobenius $T$-graded system in $\mathcal{A}_G^{\sigma;\tau}$.

□

Given $\mathcal{L}$ to be a Frobenius $T$-graded system in the category $\mathcal{A}_G^{\sigma;\tau}$, we now proceed to provide it with a structure of a Turaev $T$-crossed systems in $\mathcal{A}_G^{\sigma;\tau}$.

For defining a crossing let us define a map $\widetilde{\varphi} : G \to \mathrm{Mor}(\mathcal{L})$ by linearly extending the map given below. For $g \in G$, such that $\widetilde{t}(g) = \beta$, we set:

$$\widetilde{\varphi}(g)(f) := \widetilde{\varphi}_g(f) \otimes g = \tau(\alpha, \beta)(g \otimes f) \tag{3.39}$$

where $\widetilde{t}(f) = \alpha$ and $\widetilde{\varphi}_g = \widetilde{\varphi}(g)$. We first show that $\widetilde{\varphi}$ is a map satisfying the condition which corresponds to a homomorphism of groups.

**Proposition 3.6.6** *The map $\widetilde{\varphi} : G \to \mathrm{Mor}(\mathcal{L})$ satisfies*

$$\widetilde{\varphi}_{gh} = \widetilde{\varphi_g} \circ \widetilde{\varphi_h}.$$

PROOF: For any $f, g, h \in G$, consider

$$\begin{aligned}
\varphi_{gh}(f) * gh &= \tau(f, gh)\Big(gh * f\Big) \\
&= \tau(f, gh)\, \theta(g, h)^{-1}\Big((g * h) * f\Big) \\
&= \theta(g, h)^{-1}\, \tau(f, gh)\, \sigma(g, h, f)\Big(g * (h * f)\Big) \\
&= \theta(g, h)^{-1}\, \tau(f, gh)\, \sigma(g, h, f)\, \tau(f, h)^{-1}\Big(g * \big(\varphi_h(f) * h\big)\Big) \\
&= \theta(g, h)^{-1}\, \tau(f, gh)\, \tau(f, h)^{-1}\, \sigma(g, h, f)\, \sigma(g, hfh^{-1}, h)^{-1}\Big(\big(g * \varphi_h(f)\big) * h\Big) \\
&= X\, \tau(hfh^{-1}, g)^{-1}\Big(\big(\varphi_g\varphi_h(f) * g\big) * h\Big) \\
&= Y\, \sigma(ghfh^{-1}g^{-1}, g, h)\Big(\varphi_g\varphi_h(f) * \big(g * h\big)\Big) \\
&= Z\, \Big(\varphi_g\varphi_h(f) * (gh)\Big).
\end{aligned}$$

where the coefficients $X$, $Y$ and $Z$ are given as :

$$X = \theta(g, h)^{-1}\tau(f, gh)\tau(f, h)^{-1}\sigma(g, h, f)\sigma(g, hfh^{-1}, h)^{-1} \ ;$$

$$Y = \theta(g, h)^{-1}\tau(f, gh)\tau(hfh^{-1}, g)\tau(f, h)^{-1}\sigma(g, h, f)\sigma(g, hfh^{-1}, h)^{-1} \ ;$$

and

$$Z = \tau(f, gh)\tau(f, h)^{-1}\tau(hfh^{-1}, g)^{-1}\sigma(g, h, f)\sigma(g, hfh^{-1}, h)^{-1}\sigma(ghfh^{-1}g^{-1}, g, h).$$

We want to show the coefficient $Z$ is identity. Combining the terms $\tau(f, gh)$, $\tau(f, h)^{-1}$ and $\sigma(g, h, f)$ in the expression of $Z$ and using equation (3.28), $Z$ becomes

$$\sigma(g, f, h)\tau(f, g)\tau(hfh^{-1}, g)^{-1}\sigma(f, g, h)^{-1}\sigma(g, hfh^{-1}, h)^{-1}\sigma(ghfh^{-1}g^{-1}, g, h).$$

Since $G$ is abelian, $Z$ reduces to:

$$\sigma(g,f,h)\tau(f,g)\tau(f,g)^{-1}\sigma(f,g,h)^{-1}\sigma(g,f,h)^{-1}\sigma(f,g,h)$$

and then every term cancels out so that $Z$ becomes identity.

□

Let us call a morphism $\zeta \in \mathrm{Mor}(\mathcal{L})$ to be an algebra morphism if it respects the multiplications and the unit of $\mathcal{L}$. We claim each $\widetilde{\varphi}_g$ is an algebra morphism in the following sense:

**Proposition 3.6.7** *For each $g \in G$, $\widetilde{\varphi}_g \in \mathrm{Mor}(\mathcal{L})$ is an algebra morphism.*

PROOF: It suffices to show that

$$\widetilde{\varphi}_g(f \otimes f^{'}) = \widetilde{\varphi}_g(f) \otimes \widetilde{\varphi}_g(f^{'})$$

for $f, f^{'} \in G$. Consider

$$\begin{aligned}
\varphi_g(f * f^{'}) * g &= \theta(f,f^{'})\Big(\varphi_g(ff^{'}) * g\Big) \\
&= \theta(f,f^{'})\,\tau(ff^{'},g)\Big(g * ff^{'}\Big) \\
&= \tau(ff^{'},g)\Big(g * (f * f^{'})\Big) \\
&= \sigma(g,f,f^{'})^{-1}\,\tau(ff^{'},g)\Big((g * f) * f^{'}\Big) \\
&= \sigma(g,f,f^{'})^{-1}\,\tau(ff^{'},g)\,\tau(f,g)^{-1}\Big((\varphi_g(f) * g) * f^{'}\Big) \\
&= \sigma(g,f,f^{'})^{-1}\,\tau(ff^{'},g)\,\tau(f,g)^{-1}\,\sigma(gfg^{-1},g,f^{'})\Big(\varphi_g(f) * (g * f^{'})\Big) \\
&= X\Big(\varphi_g(f) * (\varphi_g(f^{'}) * g)\Big) \\
&= Y\Big((\varphi_g(f) * \varphi_g(f^{'})) * g\Big).
\end{aligned}$$

where the coefficients $X$ and $Y$ are

$$X = \sigma^{-1}(g,f,f^{'})\sigma(gfg^{-1},g,f^{'})\tau(ff^{'},g)\tau(f,g)^{-1}\tau(f^{'},g)^{-1}.$$

and since $G$ is abelian, we have

$$\begin{aligned} Y &= \sigma^{-1}(g,f,f^{'})\sigma(gfg^{-1},g,f^{'})\sigma(gfg^{-1},gf^{'}g^{-1},g)^{-1}\tau(ff^{'},g)\tau(f,g)^{-1}\tau(f^{'},g)^{-1} \\ &= \sigma^{-1}(g,f,f^{'})\sigma(f,g,f^{'})\sigma(f,f^{'},g)^{-1}\tau(ff^{'},g)\tau(f,g)^{-1}\tau(f^{'},g)^{-1} \\ &= 1 \quad \text{using equation (3.29).} \end{aligned}$$

This completes the proof. □

Let us now define the crossing $\varphi : T \to \mathrm{Mor}(\mathcal{L})$ for the system $\mathcal{L}$. Consider the following diagram:

$$\begin{tikzcd} G \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\widetilde{t}"'] & \mathrm{Mor}(\mathcal{L}) \\ T \arrow[ur, dotted, "\varphi"'] & \end{tikzcd}$$

For $\varphi$ to be well defined it suffices to show that $\varphi\widetilde{t}|_Z = \widetilde{\varphi}|_Z = \mathrm{Id}$ where $Z = \{g \in G|\ \widetilde{\varphi}_g = I\}$. Thus we have the following proposition:

**Proposition 3.6.8** *Given that $(\partial\theta)_2 = \tau|_Z$, the crossing $\varphi : T \to \mathrm{Mor}(\mathcal{L})$ becomes a well defined map.*

Proof: Consider $g \in Z \subseteq G$ such that $\widetilde{t}(g) = \beta$. Since $g \in Z$, $\widetilde{\varphi}_g = 1_g$. Then for any $f \in G$ with $\widetilde{t}(f) = \alpha$, equation (3.39) becomes $f \otimes h = \tau(\alpha,\beta)\, h \otimes f$

$$\begin{aligned} &\Rightarrow \quad \theta(f,h)fh = \theta(h,f)\tau(\alpha,\beta)hf \\ &\Rightarrow \quad \theta(f,h)\theta(h,f)^{-1} = \tau(\alpha,\beta) \end{aligned}$$

which is essentially the given condition that $\partial\theta = \tau|_Z$. □

Now by definition (3.39),

$$\widetilde{\varphi}_g(f) \otimes g = \tau(\alpha,\beta)(g \otimes f).$$

Then by simple calculations mainly using (3.34) and (3.39), we can rearrange this formula to get:

$$g^{-1} \otimes \widetilde{\varphi}_g(f) = X_{f,g}(f \otimes G^{-1}), \tag{3.40}$$

where the coefficient $X_{f,g}$ is

$$X_{f,g} = \theta(g^{-1}, g)\theta(g, g^{-1})^{-1}\tau(f, g^{-1})\sigma(g, f, g^{-1})\sigma(f, g, g^{-1})^{-1}\sigma(g^{-1}, g, fg^{-1})^{-1}.$$

By abuse of notation, we will identify $\widetilde{\varphi}$ with $\varphi$ and keep working on the group $G$ for proving the results because we can easily extend it onto $T$.

Note that the pairing $\phi$ for the system $\mathcal{L}$ is given by the same rule as in equation (3.24). We have the following result:

**Proposition 3.6.9** *The pairing $\phi$ for the system $\mathcal{L}$ preserves the crossing $\varphi$.*

Proof: Since $G$ is abelian, we have for any $g, h \in G$:

$$\begin{aligned}
\phi\big(\widetilde{\varphi_g(f)}, \widetilde{\varphi}_g(f^{'})\big) &= \theta(gfg^{-1}, gf^{'}g^{-1}) \\
&= \theta(f, f^{'}) \\
&= \phi(f, f^{'}).
\end{aligned}$$

□

Let $h = fgf^{-1}g^{-1}$, for $f, g \in G$. Let

$$\widetilde{b_f} : h \otimes \widetilde{\varphi}_g(\_) : Kf \longrightarrow Kf,$$

$$\widetilde{c_g} : \widetilde{\varphi}_{f^{-1}}(h \otimes \_) : Kg \longrightarrow Kg$$

be such that $\widetilde{b_f} : f \mapsto kf$ and $\widetilde{c_g} : g \mapsto k^{'}g$ for some $k, k^{'} \in K$. Then $\mathrm{Tr}\ \widetilde{b_f} = k$ and $\mathrm{Tr}\ \widetilde{c_g} = k^{'}$. For the last axiom of a Turaev crossed system related to trace, we need to show that the linear extensions of $\widetilde{b_f}$ and $\widetilde{c_g}$ to components of $\mathcal{L}$ have the same trace. It will be suffices to show that $k = k^{'}$.

**Proposition 3.6.10** *The crossing $\varphi$ satisfies the trace axiom for a Turaev $T$-crossed system in $\mathcal{A}_G^{\sigma,\tau}$ if the following condition is true:*

$$\tau(\alpha, \beta)\,\theta(f^{-1}, f)\,\sigma(1, \alpha, \beta)\,\sigma(1, \alpha^{-1}, \alpha)\,\sigma(\alpha, \alpha^{-1}, \alpha) = \theta(f, f^{-1})\,\sigma(1, \beta, \alpha)\tau(1, \alpha), \quad (3.41)$$

*for $f, g \in G$.*

*[ Note that if $\tau = (\partial\theta)_2$, then we can combine $\theta(f, f^{-1})$ and $\theta(f^{-1}, f)$ and use $\tau(\alpha, \alpha^{-1}) = \theta(f, f^{-1})\theta(f^{-1}, f)^{-1}$.]*

PROOF: Let $f, g$ be in $G$ and $h = fgf^{-1}g^{-1}$ such that $\alpha, \beta, \gamma$ are their respective images under $\widetilde{t}$. Then consider the following equalities:

$$\begin{aligned}
k(f \otimes g) &= (kf) \otimes g \\
&= \big(h \otimes (\widetilde{\varphi}_g(f)\big) \otimes g \\
&= \sigma(\gamma, \beta\alpha\beta^{-1}, \beta)\Big(h \otimes \big(\widetilde{\varphi}_g(f) \otimes g\big)\Big) \\
&= \sigma(\gamma, \beta\alpha\beta^{-1}, \beta)\ \tau(\alpha, \beta)\Big(h \otimes \big(g \otimes f\big)\Big) \\
&= \tau(\alpha, \beta)\ \sigma(\gamma, \beta\alpha\beta^{-1}, \beta)\ \sigma(\gamma, \beta, \alpha)^{-1}\Big(\big(h \otimes g\big) \otimes f\Big) \\
&= X\Big(f \otimes \widetilde{\varphi}_{f^{-1}}(h \otimes g)\Big) \\
&= X\Big(f \otimes k'g\Big).
\end{aligned}$$

where the coefficient $X$ using (3.40) is

$$\begin{aligned}
= \ & \tau(\alpha, \beta)\ \tau(\alpha\beta\alpha^{-1}\beta^{-1}, \alpha)^{-1}\ \sigma(\alpha\beta\alpha^{-1}\beta^{-1}, \beta\alpha\beta^{-1}, \beta)\ \sigma(\alpha\beta\alpha^{-1}\beta^{-1}, \beta, \alpha)^{-1}\theta(f, f^{-1})^{-1} \\
& \theta(f^{-1}, f)\ \sigma(\alpha^{-1}, \alpha\beta\alpha^{-1}\beta^{-1}, \alpha)^{-1}\ \sigma(\alpha\beta\alpha^{-1}\beta^{-1}, \alpha^{-1}, \alpha)\ \sigma(\alpha, \alpha^{-1}, \alpha\beta\alpha^{-1}\beta^{-1}\alpha).
\end{aligned}$$

Since $G$ is abelian, $X$ becomes:

$$\begin{aligned}
= \ & \tau(\alpha, \beta)\ \tau(1, \alpha)^{-1}\ \sigma(1, \alpha, \beta)\ \sigma(1, \beta, \alpha)^{-1}\theta(f, f^{-1})^{-1}\theta(f^{-1}, f)\ \sigma(\alpha^{-1}, 1, \alpha)^{-1} \\
& \sigma(1, \alpha^{-1}, \alpha)\ \sigma(\alpha, \alpha^{-1}, 1).
\end{aligned}$$

Now using the given condition $X$ becomes 1. This completes our proof. □

Note that in general the condition (3.31) is only a sufficient condition. It becomes a necessary condition if $\tilde{t} : G \to T$ is an injection. Let us define a $3_\theta$-*cocycle*. Given a 2-cochain $\theta$ of an abelian group $G$, and $(\sigma, \tau)$ an abelian 3-cocycle, we say it is a $3_\theta$-*cocycle*, if it satisfies the condition (3.41), that is, for any $f, g \in T$,

$$\tau(\alpha,\beta)\,\theta(f^{-1},f)\,\sigma(1,\alpha,\beta)\,\sigma(1,\alpha^{-1},\alpha)\,\sigma(\alpha,\alpha^{-1},\alpha)=\theta(f,f^{-1})\,\sigma(1,\beta,\alpha)\tau(1,\alpha).$$

Thus we are now in a position to formulate a Turaev $T$-crossed system in $\mathcal{A}_G^{\sigma,\tau}$.

**Theorem 3.6.11** *Given that $(H,\pi,t,u)$ is a crossed module such that $^{-}: H \twoheadrightarrow G$ is a surjective homomorphism of $H$ onto abelian group $G$ with the diagram (3.33) being commutative. Let $(\sigma,\tau)$ be a $3_\theta$-cocycle of $T$. If*

*(i)* $d\theta = \sigma|_H$

*(ii)* $(\partial\theta)_2 = \tau|_Z$

*then the collection $\mathcal{L} = \{L_\alpha | \alpha \in T\}$ forms a Turaev $T$-crossed system in $\mathcal{A}_G^{\sigma,\tau}$.*

Proof: Proof of the theorem follows from Theorem 3.6.5 and Propositions 3.6.6 to 3.6.10. □

**Example.** Let $G = T = \mathbb{Z}/2$ with generator $t$ and $M = \mathbb{C}^\times$. Then $H^3_{\text{ab}}(\mathbb{Z}/2,\mathbb{C}^\times) \cong \mathrm{Hom}\,(\mathbb{Z}/4,\mathbb{C}^\times)$. Let $a \in \{\pm1,\pm\iota\}$. For $x, y, z \in T$, define $\sigma$ as:

$$\sigma(x,y,z)=\begin{cases} a^{2x} & ;\, y=z=t \\ 1 & ;\, \text{otherwise}\end{cases} \tag{3.42}$$

Let us set $\tau$ as:

$$\begin{aligned}\tau(1,1) &= 1\\ \tau(1,t) &= 1\\ \tau(t,1) &= 1\\ \tau(t,t) &= a\end{aligned}$$

Finally, for any $x \in T$, let us us set $q(x) = a^x$. Then $q(x) = q(x^{-1})$, since $x = x^{-1}$ in $\mathbb{Z}/2$. Moreover,

$$q(xyz)q(x)q(y)q(z) = a^{x^2y^2z^2} = 1,$$

$$q(yz)q(x)q(zx)q(xy) = a^{x^2y^2z^2} = 1.$$

Thus $q(x) = a^x$ does define a quadratic function on $T$. One can also easily check that (3.27) and (3.28), (3.29) are satisfied. Hence, $(\sigma, \tau)$ is an abelian 3-cocycle. Note that for $a = 1$, $\mathcal{A}^{\sigma,\tau}$ is the category of $\mathbb{Z}/2$-graded spaces. For $a = -1$, $\mathcal{A}^{\sigma,\tau}$ is the category of super vector spaces.

Note that $\mathcal{C}_\theta[G]$ is Frobenius but not Turaev in $\mathcal{A}_G^{\sigma,\theta}$ defined above. It will be interesting to find other examples of a Turaev system in $\mathcal{A}_G^{\sigma,\theta}$.

# Chapter 4

# Coloured Quantum groups

In this chapter we work with algebraic groups and group schemes. Like in the last chapter, we shall define $\mathcal{G}$-coalgebras and $\mathcal{G}$-algebras, but now $\mathcal{G}$ is a group scheme. Inspired by Ohtsuki's definition of a coloured quantum group [Oht93], we shall generalise the concept to a Hopf $\mathcal{G}$-coalgebra. In the discrete case, Virelizier has defined a Hopf group coalgebra [Vir02] whereas Ohtsuki defines a similar object but he calls it a coloured quantum group. Further we shall define a crossed structure on Hopf $\mathcal{G}$-coalgebras. This work is inspired by Turaev, [Tur99]. We then discuss quasitriangular structures on a Hopf $\mathcal{G}$-coalgebra. Finally we construct quantum double of a Hopf $\mathcal{G}$-coalgebra. This part of the thesis is inspired by the work of Zunino, [Zun04a].

We shall work with a group scheme $\mathcal{G}$ over the ground field $\mathbb{K}$. We think of it as a Zariski topological space $\mathcal{G}$ together with a structure sheaf of algebras $\mathcal{O}_{\mathcal{G}}$ on $\mathcal{G}$. The multiplication is $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, the inverse is $\iota : \mathcal{G} \to \mathcal{G}$, the identity is $e : p \to \mathcal{G}$, where $p$ is the spectrum of $\mathbb{K}$ (the point) and the conjugation is $c : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ such that $(g, h) \mapsto hgh^{-1}$ for $g, h \in \mathcal{G}$. We will denote this action by ${}^{h}g \mapsto hgh^{-1}$. Note that $e \in \mathcal{G}(\mathbb{K})$. If we denote $1_{\mathbb{S}} \in \mathcal{G}(\mathbb{S})$ as the unit element of each group $\mathcal{G}(\mathbb{S})$ for a $\mathbb{K}$-algebra $\mathbb{S}$, then $e = 1_{\mathbb{K}}$ is the unit of $\mathcal{G}(\mathbb{K})$.

## 4.1 $\mathcal{G}$-coalgebras

In chapter 3, we have discussed $G$-coalgebras in a symmetric monoidal category when $G$ is a discrete group. We shall now discuss the case when $\mathcal{G}$ is a group scheme. A

$\mathcal{G}$-coalgebra is a quasicoherent sheaf $C$ on $\mathcal{G}$ with two structure maps given by:

$$\Delta : \mu^* C \to C \boxtimes C \text{, and}$$

$$\epsilon : e^*(C) \to \mathcal{O}_p$$

that must be morphisms of quasicoherent sheaves on $\mathcal{G} \times \mathcal{G}$ and $p$ correspondingly. The box tensor product $\boxtimes$ denotes that the tensor product sheaf is over $\mathcal{G} \times \mathcal{G}$. Note that $e^*(C) = C_e = C_{1_{\mathbb{K}}}$, all of them denoting the stalk of $C$ at unity. In future we shall refer all these simply by $C_1$. The structures introduced above satisfy respectively the axiom of coassociativity and the axiom of counity. Coassociativity looks nearly usual:

$$(\Delta \boxtimes I) \circ (\mu \times 1)^* \Delta = (I \boxtimes \Delta) \circ (1 \times \mu)^* \Delta.$$

If $\mu_2 : \mathcal{G} \times \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a double multiplication then the coassociativity can be expressed as the commutativity of the following diagram:

$$\begin{tikzcd}
\mu_2^* C \arrow[r, "(\mu\times 1)^*\Delta"] \arrow[d, "(1\times\mu)^*\Delta"'] & \mu^* C \boxtimes C \arrow[d, "\Delta\boxtimes \mathrm{id}_C"] \\
C \boxtimes \mu^* C \arrow[r, "id_C\boxtimes\Delta"'] & C \boxtimes C \boxtimes C
\end{tikzcd}$$

The two axioms of counity can be expressed using $r_e$, the restriction to the stalk at $e$ and the two projection maps $\pi_1, \pi_2 : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$. In particular, the counity can be thought of as a linear map $C_e \to \mathbb{K}$. Let us define the morphisms

$$\alpha_1 : \mu^* C \xrightarrow{\Delta} C \boxtimes C \xrightarrow{\mathrm{id}_C \boxtimes r_e} C \boxtimes e_*(C_e) \xrightarrow{\mathrm{id}_C \boxtimes e_* \epsilon} C \boxtimes e_*(\mathbb{K}) \xrightarrow{\cong} \pi_1^* C,$$

$$\alpha_2 : \mu^* C \xrightarrow{\Delta} C \boxtimes C \xrightarrow{r_e \boxtimes \mathrm{id}_C} e_*(C_e) \boxtimes C \xrightarrow{e_* \epsilon \boxtimes \mathrm{id}_C} e_*(\mathbb{K}) \boxtimes C \xrightarrow{\cong} \pi_2^* C.$$

Here $e : \{p\} \to \mathcal{G}$ is the natural embedding and $e_*(C_e)$ is the pushforward of the stalk at $e$. Notice that $\mu^* C|_{\mathcal{G}\times e} = \pi_1^* C|_{\mathcal{G}\times e}$. Finally, the counity axiom says that both $\alpha_1|_{\mathcal{G}\times e}$ and $\alpha_2|_{e\times\mathcal{G}}$ are identity maps.

Suppose $(C, \mu_C, \epsilon_C)$ and $(D, \mu_D, \epsilon_D)$ are $\mathcal{G}$-coalgebras. A morphism $f$ between two $\mathcal{G}$-coalgebras is a morphism of quasicoherent sheaves on $\mathcal{G}$ such that it preserves the structure maps. This demands the commutativity of the following two diagrams:

$$\begin{tikzcd}
\mu^*(C) \arrow[r, "\Delta_C"] \arrow[d, "\mu^*(f)"'] & C \boxtimes C \arrow[d, "f \boxtimes f"] \\
\mu^*(D) \arrow[r, "\Delta_D"'] & D \boxtimes D
\end{tikzcd}
\qquad
\begin{tikzcd}
e^*(C) \arrow[r, "\epsilon_C"] \arrow[d, "e^*(f)"'] & \mathcal{O}_p \\
e^*(D) \arrow[ur, "e^*(f)"'] &
\end{tikzcd}$$

The composition of morphisms is defined in the obvious way. Let us call the category of $\mathcal{G}$-coalgebras by $\mathcal{G}$-$Coalg$.

It may be interesting to discuss what a $\mathcal{G}$*-algebra* is in this context. Note that since the structure sheaf $\mathcal{O}_X$ is a sheaf of algebras, so $\mathcal{O}_X^\circ$ is a cosheaf of coalgebras. The trivial $\mathcal{G}$-coalgebra is $\mathcal{O}_{\mathcal{G}}$. Its finite dual is the trivial $\mathcal{G}$-algebra. It is $\mathcal{O}_{\mathcal{G}}^\circ$, the cosheaf of coalgebras on $\mathcal{G}$. A $\mathcal{G}$-algebra $A$ should be a cosheaf of $\mathcal{O}_{\mathcal{G}}^\circ$-comodules with multiplication and unit as:

$$m : A \boxtimes A \to \mu^* A \,,$$

$$\eta : \mathcal{O}_p \to e^*(A)$$

which must be morphisms of cosheaves of coalgebras on $\mathcal{G} \times \mathcal{G}$ and $p$ respectively. These structures would satisfy associativity axiom and unit axiom respectively. This discussion makes sense provided the appropriate notions of a pullback and cotensor product for cosheaves exists. We will pursue this discussion in the future publications.

We know from chapter 2 that a group scheme defines a functor from category $\mathrm{Alg}_{\mathbb{K}}$ of $\mathbb{K}$-algebras to the category Grp of groups. So, given a group scheme $\mathcal{G}$ and a $\mathbb{K}$-algebra $\mathbb{S}$, $\mathcal{G}(\mathbb{S})$ defined as $\mathrm{Mor}_{\mathrm{Sch}}\big(\mathrm{Spec}\,(\mathbb{S}), \mathcal{G}\big)$ is a group. We wish to define a $\mathcal{C}$-structure on $\mathcal{G}$ for a general category $\mathcal{C}$ with a functor $\varphi$ from $\mathcal{C}$ to the category Grp of groups.

**Definition 4.1.1** *A $\mathcal{C}$-structure on $\mathcal{G}$ is a functor $\alpha : \mathrm{Alg}_{\mathbb{K}} \to \mathcal{C}$ such that the following* *diagram of functors*

$$\begin{tikzcd} & \mathcal{C} \arrow[dd, "\varphi"] \\ \mathrm{Alg}_{\mathbb{K}} \arrow[ur, "\alpha"] \arrow[dr, "\mathcal{G}"'] & \\ & \mathrm{Grp} \end{tikzcd}$$

*is naturally commutative. This means there exists a natural isomorphism or isomorphism of functors between $\varphi \circ \alpha$ and $\mathcal{G}$.*

Note that the natural isomorphism is not a part of the $\mathcal{C}$-structure.

**Examples.** We first set and fix categories $\mathcal{C}_0$ and $\mathcal{C}$ and then Proposition 4.1.2 will provide an example for a $\mathcal{C}$-structure.

(i) Let us define a category $\mathcal{C}_0$ as follows. An object in this category is a pair $(V, G)$ where $V = \{V_g\}_{g \in G}$ is a $G$-graded vector space and $G$ a group. A morphism between objects $(V, G)$ and $(W, H)$ is a pair $(\alpha_0, \alpha)$ where $\alpha_0 : G \to H$ is a group morphism and $\alpha = \{\alpha_g\}_{g \in G}$ is a collection of linear maps. For each $g \in G$, $\alpha_g$ is a map from $V_g$ to $W_{\alpha_0(g)}$.

(ii) Define another category $\mathcal{C}$ as follows. An object in this category is a triple $(V, G, \mathbb{S})$ where $V = \{V_g\}_{g \in G}$ is a $G$-graded $\mathbb{S}$-module, $G$ a group, and $\mathbb{S}$ is a $\mathbb{K}$-algebra. A morphism between objects $(V, G, \mathbb{S})$ and $(W, H, \mathbb{S}')$ is a triple $(\alpha_0, \alpha_1, \alpha)$ where $\alpha_0 : G \to H$ is a group morphism, $\alpha_1 : \mathbb{S} \to \mathbb{S}'$ is a $\mathbb{K}$-algebra morphism and $\alpha = \{\alpha_g\}_{g \in G}$ is a collection of $\mathbb{S}'$-linear maps. For each $g \in G$, $\alpha_g$ is a map from $\mathbb{S}' \otimes_{\mathbb{S}} V_g$ to $W_{\alpha_0(g)}$.

Using the definition for $\mathcal{C}$, we have the following result:

**Proposition 4.1.2** *A quasicoherent sheaf of $\mathbb{K}$-algebras $\mathcal{F}$ on a group scheme $\mathcal{G}$ gives a $\mathcal{C}$-structure on $\mathcal{G}$.*

Proof: Suppose $\mathbb{S}$ is a $\mathbb{K}$-algebra. Then the stalks of $\mathcal{F}$ on elements of the group $\mathcal{G}(\mathbb{S})$ form a $\mathcal{G}(\mathbb{S})$-graded $\mathbb{S}$-module. At the level of objects, we have the following naturally commutative diagram,

$$\begin{tikzcd}[column sep=small] & \Big(\mathcal{F}_x, \mathcal{G}(\mathbb{S}), \mathbb{S}\Big)_{x\in\mathcal{G}(\mathbb{S})} \arrow[d, mapsto, "\varphi"] \\ \mathbb{S} \arrow[ur, mapsto, "\alpha"] \arrow[dr, mapsto, "\mathcal{G}"'] & \\ & \mathcal{G}(\mathbb{S}) \end{tikzcd}$$

and at the level of morphism, we have :

$$\begin{tikzcd}[column sep=small] & \Big(\mathbb{S}' \otimes_{\mathbb{S}} \mathcal{F}_x \to \mathcal{F}_{\mathcal{G}(\pi)(x)}\Big)_{x\in\mathcal{G}(\mathbb{S})} \arrow[d, mapsto, "\varphi"] \\ \big(\mathbb{S} \xrightarrow{\pi} \mathbb{S}'\big) \arrow[ur, mapsto, "\alpha"] \arrow[dr, mapsto, "\mathcal{G}"'] & \\ & \Big(\mathcal{G}(\mathbb{S}) \xrightarrow{\mathcal{G}(\pi)} \mathcal{G}(\mathbb{S}')\Big). \end{tikzcd}$$

□

In particular consider the affine case: For a $\mathbb{K}$-algebra $\mathbb{S}$, and an affine group scheme $\mathcal{G}$, let $x \in \mathcal{G}(\mathbb{S})$. Then $x$ is a ring homomorphism. Let $x$ be a morphism from algebra of functions $A$ to $\mathbb{S}$. Then a quasicoherent sheaf $\mathcal{F}$ can simply be taken as a left $A$ module $M$. The stalk of $\mathcal{F}$ at $x$ is the $\mathbb{S}$-module, $\mathcal{F}_x = \mathbb{S} \otimes_A M$. Choose a $\mathbb{K}$-algebra morphism $\pi : \mathbb{S} \to \mathbb{S}'$, then the point $\mathcal{G}(\pi)(x)$ is a ring homomorphism from $A$ to $\mathbb{S}'$ :

$$\mathcal{G}(\pi)(x) : A \xrightarrow{x} \mathbb{S} \xrightarrow{\pi} \mathbb{S}'.$$

The stalk of $\mathcal{F}$ at point $\mathcal{G}(\pi)(x)$ is given as :

$$\mathcal{F}_{\mathcal{G}(\pi)(x)} = \mathbb{S}' \otimes_A M \cong \mathbb{S}' \otimes_{\mathbb{S}} \mathbb{S} \otimes_A M = \mathbb{S}' \otimes_{\mathbb{S}} \mathcal{F}_x.$$

Then the map between stalk at $x$ and stalk at $\mathcal{G}(\pi)(x)$ is $f \longmapsto 1 \otimes_{\mathbb{S}} f$ for $f \in \mathcal{F}_x$.

**Example.** We define yet another category $\mathcal{C}_C$ where the subscript $C$ refers to *coalgebra*. An object in this category is a triple $(V, G, \mathbb{S})$ where $V = \{V_g\}_{g\in G}$ is a collection of $\mathbb{S}$-modules forming a $G$-coalgebra in the sense of (3.1). A morphism between objects

$(V, G, \mathbb{S})$ and $(W, H, \mathbb{S}^{'})$ is a triple $(\rho_0, \rho_1, \rho)$ where $\rho_0 : G \to H$ is a group homomorphism, $\rho_1 : \mathbb{S} \to \mathbb{S}^{'}$ is a $\mathbb{K}$-algebra morphism and $\rho$ is a collection of $\mathbb{S}^{'}$-module maps preserving the $G$-coalgebra structures, i.e. $\rho : \mathbb{S}^{'} \otimes V \to \rho_0^*(W)$ is a morphism of $G$-coalgebras over $\mathbb{S}^{'}$. Note that for each $g \in G$, $\mathbb{S}^{'} \otimes_{\mathbb{S}} V_g$ is an $\mathbb{S}^{'}$-module, and $\{\mathbb{S}^{'} \otimes_{\mathbb{S}} V_g\}_{g \in \mathcal{G}}$ forms a $\mathcal{G}$-coalgebra over $\mathbb{S}^{'}$. Again for each $g \in G$, $\left(\rho_0^*(W)\right)_g = W_{\rho_0(g)}$ and the collection $\rho_0^*(W) = \left\{W_{\rho_0(g)}\right\}_{g \in G}$ forms a $G$-coalgebra of $\mathbb{S}^{'}$-modules. Thus $\rho = \{\rho_g : \mathbb{S}^{'} \otimes_{\mathbb{S}} V_g \longrightarrow W_{\rho_0(g)}\}_{g \in \mathcal{G}}$ is the required collection of homomorphisms.

Given a $\mathcal{G}$-coalgebra $\mathcal{F}$ on a group scheme $\mathcal{G}$, we have a functor $\alpha : \mathrm{Alg}_{\mathbb{K}} \to \mathcal{C}_C$ taking an object $\mathbb{S}$ to $(\mathcal{F}_x, \mathcal{G}(\mathbb{S}))_{x \in \mathcal{G}(R)}$, and a morphism $\pi : \mathbb{S} \to \mathbb{S}^{'}$ to $(\mathcal{F}_x \to \mathcal{F}_{\mathcal{G}(\pi)(x)})_{x \in \mathcal{G}(\mathbb{S})}$. There is another functor $\varphi : \mathcal{C}_C \to \mathrm{Grp}$ taking an object $(V, G)$ to $G$, and a morphism $f = (f_1, f_2) : (V, G) \to (W, H)$ to the group morphism $f_2 : G \to H$. Using this data, we have the following result.

**Proposition 4.1.3** *A $\mathcal{G}$-coalgebra on a group scheme $\mathcal{G}$ gives a $\mathcal{C}_C$-structure on $\mathcal{G}$.*

PROOF: Let $\mathbb{S}$ be a $\mathbb{K}$-algebra and let $\mathcal{F}$ be a $\mathcal{G}$-coalgebra. We claim that the collection of stalks of $\mathcal{F}$ on elements of the group $\mathcal{G}(\mathbb{S})$ form a $\mathcal{G}(\mathbb{S})$-coalgebra. Let us also denote this collection by $\mathcal{F} = \{\mathcal{F}_x\}_{x \in G}$, where $G = \mathcal{G}(\mathbb{S})$. Now the comultiplication $\Delta : \mu^*\mathcal{F} \to \mathcal{F} \boxtimes \mathcal{F}$ of the sheaf $\mathcal{F}$ at the level of a stalk at $(g, h) \in G \times G$ looks like:

$$\Delta_{g,h} : \mathcal{F}_{gh} \to \mathcal{F}_g \boxtimes \mathcal{F}_h,$$

and the counit $\epsilon : e^*(\mathcal{F}) \to \mathcal{O}_p$ at the level of a stalk at $\{p\}$ is the following map:

$$\epsilon : \mathcal{F}_1 \to \mathbb{S}$$

Note that the stalk of $\mu^*\mathcal{F}$ at $(g, h)$ is $\mathcal{F}_{\mu(g,h)} = \mathcal{F}_{gh}$. Moreover, $\mathcal{F}_g \boxtimes \mathcal{F}_h$ is the stalk of the sheaf $\mathcal{F} \boxtimes \mathcal{F}$ at $(g, h)$. Further, $\mathcal{O}_p(p) = \mathcal{O}_{\mathrm{Spec}\,\mathbb{K}}(\mathrm{Spec}\,\mathbb{K}) = \mathbb{K}$. At the level of objects and morphism, we have the following naturally commutative diagrams,

$$
\begin{tikzcd}
 & \Big(\mathcal{F}_x, \mathcal{G}(\mathbb{S})\Big)_{x\in\mathcal{G}(\mathbb{S})} \arrow[d, mapsto, "\varphi"] \\
\mathbb{S} \arrow[ur, mapsto, "\alpha"] \arrow[dr, mapsto, "\mathcal{G}"'] & \\
 & \mathcal{G}(\mathbb{S})
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \Big(\mathbb{S}' \otimes \mathcal{F}_x \to \mathcal{F}_{\mathcal{G}(\pi)(x)}\Big)_{x\in\mathcal{G}(\mathbb{S})} \arrow[d, mapsto, "\varphi"] \\
(\mathbb{S} \xrightarrow{\pi} \mathbb{S}') \arrow[ur, mapsto, "\alpha"] \arrow[dr, mapsto, "\mathcal{G}"'] & \\
 & \Big(\mathcal{G}(\mathbb{S}) \xrightarrow{\mathcal{G}(\pi)} \mathcal{G}(\mathbb{S}')\Big).
\end{tikzcd}
$$

The details are obvious. □

**Corollary 4.1.4** *For a group scheme $\mathcal{G}$, and each $\mathbb{K}$-algebra $\mathbb{S}$, the collection of stalks $\{C_x\}_{x\in\mathcal{G}(\mathbb{S})}$ is a $\mathcal{G}(\mathbb{S})$-coalgebra in category of $\mathbb{S}$-modules. This defines a functor from $\mathcal{G}$-Coalg to $\mathcal{G}(\mathbb{S})$-Coalg$_{\mathbb{S}}$.*

PROOF: The functor $C \mapsto (C_x)_{x\in G}$ gives the required result. □

Note that in general, the functor defined in the Corollary 4.1.4 need not give an equivalence of categories. For example consider the $G$-coalgebra $\{C_g\}_{g\in\mathbb{Z}}$ such that $C_g = \mathbb{K}$ if $g = 0$ and $C_g = 0$ otherwise. Then the skyscraper sheaf supported at origin that we will recover from these stalks is not a quasicoherent sheaf over $(X, \mathcal{O}_X)$ where $X = \operatorname{Spec}\mathbb{Z}[x]$!

**Corollary 4.1.5** *If $\mathcal{G}$ is discrete reduced group scheme, then there is an equivalence of categories between $\mathcal{G}$-Coalg and $\mathcal{G}(\mathbb{S})$-Coalg for each $\mathbb{K}$-algebra $\mathbb{S}$.*

PROOF: The functor $C \mapsto (C_x)_{x\in G}$ gives the required equivalence. □

Note that these corollaries imply that the new definition of a $\mathcal{G}$-coalgebra agrees with the old one (defined in Section 3.1) for a discrete group scheme.

## 4.2 Hopf $\mathcal{G}$-coalgebras

In this section we introduce Hopf $\mathcal{G}$-coalgebra for a group scheme $\mathcal{G}$. At the level of fibres, they are Hopf group algebras which were introduced by Virelizier in [Vir02]. Corresponding to Hopf coalgebra structure, we then define $\mathcal{C}_H$-structure on a group scheme. Finally we discuss global cosections of a cosheaf.

### 4.2.1 Hopf group-coalgebras

In this section we discuss the definition of a Hopf group-coalgebra discussed by various authors including Ohtsuki [Oht93], V.Turaev [Tur99] and Virelizier [Vir02]. Let $P$ be a multiplicative group with identity element $e$. A *Hopf group-coalgebra*, (in our case a Hopf $P$-coalgebra) is a datum $(\{A_p\}, \{\Delta_{p,q}\}, \{S_p\}, \epsilon)$ with $p, q \in P$. Each $A_p$ is an $\mathbb{K}$-algebra (associative and unitary by default), the algebra morphisms $\Delta_{p,q} : A_{pq} \to A_p \otimes A_q$ form a comultiplication, the linear maps $S_p : A_p \to A_{p^{-1}}$ form an antipode, and finally the algebra morphism $\epsilon : A_e \to \mathbb{K}$ is a counit for the system. Denoting the algebra operations in $A_p$ by $m_p : A_p \otimes A_p \to A_p$ and $i_p : \mathbb{K} \to A_p$ the structure maps should satisfy the following axioms.

(1) $(\mathrm{id} \otimes \Delta_{p_2,p_3}) \circ \Delta_{p_1,p_2p_3} = (\Delta_{p_1,p_2} \otimes \mathrm{id}) \circ \Delta_{p_1p_2,p_3} : A_{p_1p_2p_3} \to A_{p_1} \otimes A_{p_2} \otimes A_{p_3}$

(2) $(\mathrm{id} \otimes \epsilon) \circ \Delta_{p,e} = \mathrm{id} : A_p \to A_p$

(3) $(\epsilon \otimes \mathrm{id}) \circ \Delta_{e,p} = \mathrm{id} : A_p \to A_p$

(4) $m_p \circ (\mathrm{id} \otimes S_{p^{-1}}) \circ \Delta_{p,p^{-1}} = i_p \circ \epsilon : A_e \to A_p$

(5) $m_p \circ (S_{p^{-1}} \otimes \mathrm{id}) \circ \Delta_{p^{-1},p} = i_p \circ \epsilon : A_e \to A_p$

(6) $\Delta_{p,q}$ and $\epsilon$ are homomorphisms of $\mathbb{K}$ algebras.

This definition has been given by Ohtsuki [Oht93] and he calls it a coloured Hopf algebra over $P$. But there are three important differences from his definition. Firstly, he demands the field to have characteristic zero and we don't. Secondly, he demands the group $P$ to be abelian. Finally, he omits axiom (6), although all his examples satisfy it. Note that axiom (6) is necessary if one wants to obtain the definition of a usual Hopf algebra in the case of a trivial group $P$. The axiom (6) is required to conclude that $(A_e, \Delta_{e,e}, S_e, \epsilon)$ is a Hopf algebra in the usual sense, which has been noted by Ohtsuki as well. For simplicity we denote the collection $(\{A_p\}, \{\Delta_{p,q}\}, \{S_p\}, \epsilon)$ by simply $(A_p)$. Ohtsuki has an example of a coloured Hopf algebra that he derives from a quantum group $U_q(sl_2)$ at a root of unity, [Oht93]..

Given a group $P$, and two Hopf $P$-coalgebras $\{A_p, \Delta^A, S^A, \epsilon^A\}$ and $\{B_p, \Delta^B, S^B, \epsilon^B\}$, we define a Hopf group coalgebra morphism $\phi$ between them as a collection $\phi = \{\phi_f : A_f \to B_f\}_{f \in P}$ of algebra morphisms which respects all the structure morphisms of the two. Explicitly, this demands the commutativity of the following diagrams:

$$\begin{tikzcd} A_{fg} \arrow[r, "\phi_{fg}"] \arrow[d, "\Delta^A_{fg}"'] & B_{fg} \arrow[d, "\Delta^B_{fg}"] \\ A_f \otimes A_g \arrow[r, "\phi_f \otimes \phi_g"'] & B_f \otimes B_g \end{tikzcd} \qquad \begin{tikzcd} A_f \arrow[r, "S^A_f"] \arrow[d, "\phi_f"'] & A_{f^{-1}} \arrow[d, "\phi_{f^{-1}}"] \\ B_f \arrow[r, "S^B_f"'] & B_{f^{-1}} \end{tikzcd}$$

$$\begin{tikzcd} A_e \arrow[r, "\epsilon^A"] \arrow[d, "\phi_e"'] & \mathbb{K} \\ B_e \arrow[ur, "\epsilon^B"'] & \end{tikzcd}$$

We have following result, which has also been discussed by A.Virelizier, and hence is of historical interest now. In fact, this result, [Vir02] is a particular case of Theorem 4.2.2 which we discuss in the next section.

**Proposition 4.2.1** *Let $(A_p)$ be a Hopf group coalgebra over a finite group $P$. Then $\mathbf{A} = \oplus_{p \in P} A_p$ is a Hopf algebra equipped with a Hopf algebra homomorphism $\phi : (\mathbb{K}P)^* \to \mathbf{A}$ whose image lies in the centre of $\mathbf{A}$.*

*Conversely, let $(\mathbf{A}, \phi)$ be a Hopf algebra and a homomorphism as above. This pair gives rise to a coloured quantum group with $A_p = A\phi(\delta_p)$ where $\delta_p \in (\mathbb{K}P)^*$ is the idempotent corresponding to $p \in P$ (delta function at $p$).*

### 4.2.2 Hopf $\mathcal{G}$-coalgebras

Let us suppose that $\mathcal{G}$ is a group scheme. Similar to the concept of a $\mathcal{C}$-structure introduced in section 4.1, one can define a $\mathcal{C}_H$ structure on $\mathcal{G}$. For this we first define the $\mathcal{C}_H$-category where the subscript $H$ refers to *Hopf group coalgebra*. An object in this category is a pair $(V, G)$ where $G$ is a group and $V$ is a Hopf $G$-coalgebra in the sense of (4.2.1). A morphism between objects $(V, G)$ and $(W, H)$ is a pair $(\alpha_0, \alpha)$ where $\alpha_0 : H \to G$ is a group homomorphism and $\alpha : V \to \alpha_0^*(V)$ is a map from $V$ to the pullback of $V$ under the map $\alpha_0$. For each $h \in H$, $\left(\alpha_0^*(V)\right)_h = W_{\alpha_0(h)}$ , thus the collection $\left\{\alpha_0^*(V)\right\}_{h \in H}$ is the same as the collection $\{W_{\tilde{h}}\}_{\tilde{h} \in H}$ as an $H$-coalgebra. Using this data, and the functor $\varphi : \mathcal{C}_H \to \mathrm{Grp}$ taking an object $(V, G)$ to $G$, and a morphism $f = (f_1, f_2) : (V, G) \to (W, H)$ to the group morphism $f_2 : G \to H$, we can define a $\mathcal{C}_H$

structure on $\mathcal{G}$ on the lines of Definition 4.1.1. We spell it out here again. A $\mathcal{C}_H$-structure on $\mathcal{G}$ is a functor $\alpha : \mathrm{Alg}_{\mathbb{K}} \to \mathcal{C}_H$ such that the following diagram of functors

$$\begin{tikzcd} & \mathcal{C}_H \arrow[dd, "\varphi"] \\ \mathrm{Alg}_{\mathbb{K}} \arrow[ur, "\alpha"] \arrow[dr, "\mathcal{G}"'] & \\ & \mathrm{Grp} \end{tikzcd}$$

is naturally commutative. This essentially requires a natural isomorphism between $\varphi \circ \alpha$ and $\mathcal{G}$. Note that naturality is not a part of the $\mathcal{C}_H$-structure. Thus we can summarize this discussion in the form of a definition for a Hopf $\mathcal{G}$-coalgebra in terms of a $\mathcal{C}$-structure. We state that a Hopf $\mathcal{G}$-coalgebra is a quasicoherent sheaf $\mathcal{A}$ over $\mathcal{G}$ of algebras together with three operations :

$$\Delta : \mu^*\mathcal{A} \to \mathcal{A} \boxtimes \mathcal{A}$$

$$\epsilon : \mathcal{A}_e \to \mathbb{K}$$

$$S : \mathcal{A} \to \iota^*\mathcal{A}$$

that must be morphisms of quasicoherent sheaves on $\mathcal{G} \times \mathcal{G}$, $p$ and $\mathcal{G}$ correspondingly. These structures are subject to the following two conditions:

(i) $(\mathcal{A}, \Delta, \epsilon)$ forms a $\mathcal{G}$-coalgebra.

(ii) $R \mapsto \big(\mathcal{A}_R, \mathcal{G}(R)\big)$ gives a $\mathcal{C}_H$-structure on $\mathcal{G}$

Note that here $R$ is a commutative $\mathbb{K}$-algebra, $\mathcal{A}_R$ is specialisation of $\mathcal{A}$ at $R$, and $\mathcal{G}(R)$ is the algebraic group of all scheme morphisms from $\mathrm{Spec}\,(R)$ to $\mathcal{G}$.

**Theorem 4.2.2** *Let $\mathcal{G}$ be an affine group scheme. Let $\mathcal{A}$ be a Hopf $\mathcal{G}$-coalgebra over $\mathcal{G}$. Then $A = \Gamma(\mathcal{G}, \mathcal{A})$ is a Hopf algebra containing $\mathbb{K}[\mathcal{G}]$ as a central Hopf subalgebra. Conversely, if $A$ is a Hopf algebra containing the global sections $\mathbb{K}[\mathcal{G}] = \Gamma(\mathcal{G}, \mathcal{O}_{\mathcal{G}})$ as a central Hopf subalgebra then the quasicoherent sheaf $\widetilde{A}$ associated to $A$ is a Hopf $\mathcal{G}$-coalgebra.*

Proof: Let $\mathcal{G} = \mathrm{Spec}\,(R)$ for a commutative ring $R$ with unit. Let $A$ be a Hopf algebra containing the global sections of $\mathcal{G}$ as a central Hopf subalgebra. Denote it by $S$. Then $A$

becomes an $S$-algebra. The sheaf $\widetilde{A}$ is the sheaf of $\mathcal{O}_{\mathcal{G}}$-modules, and hence of $S$-modules. In fact for any $f \in S$, $\widetilde{A}_f = A \otimes_S S$. The comultiplication $\Delta$ from $\mu^* \widetilde{A}$ to $\widetilde{A} \boxtimes \widetilde{A}$, is defined using the map $A \otimes_S (S \otimes_{\mathbb{K}} S) \to A \otimes_{\mathbb{K}} A$ which is given by $a \otimes (s' \otimes s'') \mapsto \sum_{(a)} a_1 s' \otimes a_2 s''$. Note that the comultiplication of $A$ maps $a$ to $\sum_{(a)} a_1 \otimes a_2$ in $A \otimes A$. The axiom for comultiplication of a Hopf $\mathcal{G}$-coalgebra follows from the axiom of comultiplication of $A$. The counit and antipode of $\widetilde{A}$ is given by the counit map respectively the antipode map of $A$. Then, it is easy to check the axioms of counit as well as antipode. Thus, as an algebra (or a sheaf of algebras) $A$ uniquely determines $\widetilde{A}$. Conversely, let $\mathcal{A}$ be a Hopf $\mathcal{G}$-coalgebra. We want to show that $\Gamma(\mathcal{G}, \mathcal{A})$ is a Hopf algebra containing the global sections of $\mathcal{G}$ as a central Hopf subalgebra. □

Virelizier has constructed two examples of a Hopf $\mathcal{G}$ coalgebra, [Vir05]. One is a Hopf $GL_n(k)$-coalgebra and the other is a Hopf $\pi$-coalgebra for $\pi = (\mathcal{C}^*)^l$ for some positive integer $l$. We show their global sections are Hopf algebras in the usual sense.

**Examples.** [Vir05]. Section 4 of this paper contains an example of a Hopf $GL_n(\mathbb{K})$-coalgebra and Sections 5 has an example of a Hopf $\pi$-coalgebra which are briefly described below.

(i) In this example, $\mathbb{K}$ is a field whose characteristics is not 2. Fix a positive integer $n$. Let $\mathcal{G} = GL_n(\mathbb{K})$ be the group of invertible $n \times n$ matrices with coefficients in $\mathbb{K}$. For $\alpha = (\alpha_{i,j}) \in GL_n(\mathbb{K})$, let $\mathcal{A}_n^\alpha$ be the $\mathbb{K}$-algebra generated by $g, x_1, \ldots, x_n, y_1, \ldots, y_n$, subject to the following relations:

$$g^2 = 1, x_1^2 = \cdots = x_n^2 = 0, gx_i = -x_i g, x_i x_j = -x_j x_i, \tag{4.1}$$

$$y_1^2 = \cdots = y_n^2 = 0, gy_i = -y_i g, y_i y_j = -y_j y_i, \tag{4.2}$$

$$x_i y_j - y_j x_i = (\alpha_{j,i} - \delta_{i,j}) g \tag{4.3}$$

where $1 \leq i, j \leq n$. Set $\alpha = (\alpha_{i,j})$, then Virelizier shows that the family $\mathcal{A}_n = \{\mathcal{A}_n^\alpha\}_{\alpha \in \mathcal{G}}$ has a structure of a crossed Hopf $\mathcal{G}$-coalgebra. The comultiplication and antipode when applied to $x_i$ gives the following :

$$\Delta_{\alpha,\beta}(x_i) = 1 \otimes x_i + \sum_{k=1}^{n} \beta_{k,i} x_k \otimes g,$$

$$S_\alpha(x_i) = \sum_{k=1}^{n} \alpha_k g x_k$$

where $\alpha = (\alpha_{i,j})$, $\beta = (\beta_{i,j})$ for $1 \leq i, j \leq n$ are any two elements in $\mathcal{G}$. Let $\mathbb{V}_n$ be the $\mathbb{C}$-algebra generated by $g, x_i, y_i, z_{i,j}, d^{\pm}$; $1 \leq i, j \leq n$, subject to the relations (4.1), (4.2) and

$$x_i y_j - y_j x_i = (z_{j,i} - \delta_{i,j})g.$$

Let $z_{i,j}$ be central in $\mathbb{V}_n$. Let $B$ be the subalgebra of $\mathbb{V}_n$ generated by $\{z_{i,j}, d^{\pm}\}$. Then $\mathbb{V}_n$ can be considered as an $B$-algebra. Let us set

$$\Delta(x_i) = 1 \otimes x_i + \sum_{k=1}^{n} x_k \otimes g z_{k,i}$$

$$\Delta(z_{i,j}) = \sum_{k=1}^{n} z_{i,k} \otimes z_{k,j}$$

$$S(x_i) = \sum_{k=1}^{n} m_{k,i} g x_k$$

$$S(z_{i,j}) = m_{i,j}$$

where $(m_{i,j}) = (z_{i,j})^{-1}$. It is easy to check that with these operations $\mathbb{V}_n$ gets a Hopf algebra structure and in fact $B$ becomes a Hopf subalgebra of $\mathbb{V}_n$. Indeed, there is a Hopf algebra isomorphism from $B$ to $\mathbb{K}[GL_n]$. Note that the antipode map for $\mathcal{A}_n^\alpha$ sends each $x_i$ to $\sum_{k=1}^{n} \alpha_{k,i} g x_k$ where as the one for $\mathbb{V}_n$ sends $x_i$ to $\sum_{k=1}^{n} m_{k,i} g x_k$, where $(m_{i,j}) = (z_{i,j})^{-1}$. We claim that they are essentially the same maps. This can be explained as follows. We can think of $z_{i,j}$ as functions on $GL_n(\mathbb{K})$. So if $z_{i,j}(\alpha) = \alpha_{i,j}$, then $m_{i,j}(\alpha) = z_{i,j}^{-1}(\alpha^{-1})$. For example in case $n = 2$, if we set

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } \alpha^{-1} = \frac{1}{|\alpha|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

We have $S_\alpha(x_1) = \alpha_{11}gx_1 + \alpha_{21}gx_2 = agx_1 + cgx_2$, where as $S(x_1) = m_{11}(\alpha)gx_1 + m_{21}(\alpha)gx_2 = z_{22}(\alpha^{-1})gx_1 - z_{21}(\alpha^{-1})gx_2 = agx_1 + cgx_2$.

(ii) Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank $l$ with Cartan matrix $(a_{i,j})$. We let $d_i$ be the coprime integers such that the matrix $(d_i a_{i,j}$ is symmetric. Let $q$ be a fixed nonzero complex number and set $q_i^2 \neq 1$ for $i = 1, 2, \cdots, l$. Set $\pi = (\mathbb{C}^*)^l$. For $\alpha = (\alpha_1, \cdots, \alpha_l) \in \pi$, let $U_q^\alpha(\mathfrak{g})$ be the $\mathbb{C}$-algebra generated by $K_i^{\pm 1}, E_i, F_i$; $1 \leq i \leq l$, subject to the following defining relations:

$$K_iK_j = K_jK_i \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1, \tag{4.4}$$

$$K_iE_j = q_i^{a_{i,j}}E_jK_i, \tag{4.5}$$

$$K_iF_j = q_i^{-a_{i,j}}F_jK_i, \tag{4.6}$$

$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{i,j}-r}E_jE_i^r = 0 \qquad \text{if } i \neq j, \tag{4.7}$$

$$\sum_{r=0}^{1-a_{i,j}} (-1)^r \begin{bmatrix} 1-a_{i,j} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{i,j}-r}F_jF_i^r = 0 \qquad \text{if } i \neq j. \tag{4.8}$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{\alpha_iK_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{4.9}$$

The formulas given by equation (4.7) and (4.8), known as Serre relations, involve the $t$-binomial coefficient $\begin{bmatrix} m \\ n \end{bmatrix}_t$ for $t = q_i$. For $t$ an indeterminate, it is given as follows:

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \frac{(t^m - t^{-m})\big(t^{(m-1)} - t^{-(m-1)}\big)\cdots\big(t^{(m-n+1)} - t^{-(m-n+1)}\big)}{(t-t^{-1})(t^2-t^{-2})\cdots(t^n-t^{-n})}$$

Virelizier has shown that the family $U_q^\pi(\mathfrak{g}) = \{U_q^\alpha(\mathfrak{g})\}_{\alpha\in\pi}$ has a structure of a crossed Hopf $\pi$-coalgebra. Note that $\{U_q^1(\mathfrak{g}), \Delta_{1,1}, \epsilon, S_1\}$ is the usual quantum group $U_q(\mathfrak{g})$.

The comultiplication and antipode when applied to $E_i$ gives the following :

$$\Delta_{\alpha,\beta}(E_i) = \beta_i E_i \otimes K_i + 1 \otimes E_i,$$

$$S_\alpha(E_i) = -\alpha_i E_i K_i^{-1}$$

where $\alpha = (\alpha_{i,j})$, $\beta = (\beta_{i,j})$; $1 \leq i, j \leq n$ are any two elements in $\mathcal{G}$. As in the previous example, let $\mathbb{U}_q(\mathfrak{g})$ be the $\mathbb{C}$-algebra generated by $K_i^{\pm 1}, E_i, F_i, Z_i^{\pm}$; $1 \leq i \leq l$, subject to the defining relations (4.4) to (4.8), and

$$E_i F_j - F_j E_i = \frac{Z_i K_i - K_i^{-1}}{q_i - q_i^{-1}}.$$

Let $Z_i^{\pm 1}$ be central in $\mathbb{U}_q(\mathfrak{g})$. Let $C$ be the subalgebra of $\mathbb{U}_q(\mathfrak{g})$ generated by $\{Z_i^{\pm 1}\}$. Then $\mathbb{U}_q(\mathfrak{g})$ can be considered as an $C$-algebra. Let us set

$$\Delta(E_i) = E_i \otimes Z_i K_i + 1 \otimes E_i$$

$$\Delta(Z_i) = Z_i \otimes Z_i$$

$$S(E_i) = -Z_i^{-1} E_i K_i^{-1}$$

$$S(Z_i) = Z_i^{-1}$$

One can easily check that both $\mathbb{U}_q(\mathfrak{g})$ and $C$ forms a Hopf algebra with these operations. And there is a Hopf algebra isomorphism from $C$ to $\mathbb{K}[\pi]$. The explanation for a slightly different antipode of $\mathbb{U}_q(\mathfrak{g})$ than of $\{U_q^\pi(\mathfrak{g})\}$ is similar to as explained in the previous example.

Let $\mathbb{G}$ be the algebraic group $\mathcal{G}(\mathbb{S})$ for a commutative ring $\mathbb{S}$ and $\mathcal{A}$ be a Hopf $\mathbb{G}$-coalgebra. Then the specialisation of $\mathcal{A}$ at $\mathbb{S}$ forms a Hopf $\mathbb{G}$-coalgebra.

**Proposition 4.2.3** *Given a group scheme $\mathcal{G}$, and $\mathcal{A}$ a Hopf $\mathcal{G}$-coalgebra, then the specialisation of $\mathcal{A}$ at $\mathbb{S}$ forms a Hopf $\mathbb{G}$-coalgebra.*

Proof: The proof follows from definition 7 (ii), of a Hopf $\mathcal{G}$-coalgebra, which says that $(\mathcal{A}_S, \mathcal{G}(\mathbb{S}))$ is a Hopf $\mathcal{G}(\mathbb{S})$-coalgebra, for any commutative $\mathbb{K}$-algebra $\mathbb{S}$. □

Let $A_{\mathcal{G}}$ be the category of Hopf $\mathcal{G}$-coalgebras over a group scheme $\mathcal{G}$ and $B_G$ the category of Hopf $G$-coalgebra over a group $G$. Let $\mathcal{A}$ be any object in $A_{\mathcal{G}}$. Then for any commutative algebra $\mathbb{S}$ over $\mathbb{K}$, its specialization $\mathcal{A}_{\mathbb{S}}$ is an object in $B_{\mathcal{G}(\mathbb{S})}$. Thus we have

**Proposition 4.2.4** *For a commutative algebra* $\mathbb{S}$ *over* $\mathbb{K}$*, there exists a functor*

$$\mathcal{F} : A_{\mathcal{G}} \to B_{\mathcal{G}(\mathbb{S})}.$$

By $\mathrm{Supp}(A_f)$ we understand *the support* of $\mathcal{A}$ at the fibre $(A_f)$, i.e., the set of all $f \in \mathbb{G}$ such that $A_f \neq 0$. We have the following result.

**Lemma 4.2.5** *The support* $\mathrm{Supp}(A_f)$ *is a closed subgroup scheme of* $\mathbb{G}$*.*

PROOF: The support is non-empty because $A_e \neq 0$ as $\epsilon : A_e \to \mathbb{K}$ is a homomorphism of algebras. The closeness under multiplication and inverse follows from the axiom (6) and (5) correspondingly. □

### 4.2.3 Crossings

So far we have defined a $\mathcal{G}$-coalgebra and a Hopf $\mathcal{G}$-coalgebra. We now set up some more structures to the list. We provide crossings to a Hopf $\mathcal{G}$-coalgebra.

Consider a Hopf $\mathcal{G}$-coalgebra $\mathcal{A}$ over a group scheme $\mathcal{G}$. We can treat $\mathcal{G}$ as a $\mathcal{G}$-variety equipped with the conjugation action of $\mathcal{G}$ on itself. This admits a $\mathcal{G}$-equivariant structure on $\mathcal{A}$ (See chapter 1). Now on the same lines of the discrete case, [Tur00], [Vir02], we say that $\mathcal{A}$ is a *crossed Hopf* $\mathcal{G}$*-coalgebra* provided it is equipped with a $\mathcal{G}$-equivariant structure

$$\varphi : \pi_1^* \mathcal{A} \to c^* \mathcal{A}$$

where $\pi_1$ is the first projection on $\mathcal{G} \times \mathcal{G}$ and $c$ is the conjugation map on $\mathcal{G}$. Note that $\varphi$ is an isomorphism of sheaves on $\mathcal{G} \times \mathcal{G}$. The crossing must satisfy the following set of axioms :

(i) $\varphi$ is a homomorphism of sheaves of algebras over $\mathcal{G} \times \mathcal{G}$.

(ii) $\varphi$ preserves the comultiplication of $\mathcal{A}$. This is given by the commutativity of the following diagram :

$$\begin{tikzcd}
(\pi_{1,2})^*(\mu_*\mathcal{A}) \arrow[r, "\varphi_{\mu_*(\mathcal{A})}"] \arrow[d, "(\pi_{1,2})^*\Delta"'] & (c_{12,3})^*(\mu_*\mathcal{A}) \arrow[d, "(c_{12,3})^*\Delta"] \\
(\pi_{1,2})^*(\mathcal{A}\boxtimes\mathcal{A}) \arrow[r, "\varphi_{\mathcal{A}\boxtimes\mathcal{A}}"] & (c_{12,3})^*(\mathcal{A}\boxtimes\mathcal{A}),
\end{tikzcd}$$

where $\pi_{1,2} : \mathcal{G}\times\mathcal{G}\times\mathcal{G} \to \mathcal{G}\times\mathcal{G}$ is the projection of the first two components, i.e. $(f,g,h) \mapsto (f,g)$ and $c_{12,3} : \mathcal{G}\times\mathcal{G}\times\mathcal{G} \to \mathcal{G}\times\mathcal{G}$ is the conjugation action : $(f,g,h) \mapsto ({}^hf, {}^hg)$ .

(iii) $\varphi$ preserves the counit of $\mathcal{A}$. For expressing this explicitly, let us define following morphisms:

$$\alpha_1 : (\pi_1)^*(\mathcal{A}) \xrightarrow{\varphi} c^*\mathcal{A} \hookrightarrow c^*\big(e_*\mathcal{A}_1\big) \xrightarrow{c^*e_*(\epsilon)} c^*e_*(\mathcal{O}_p),$$

$$\alpha_2 : (\pi_1)^*(\mathcal{A}) \hookrightarrow (\pi_1)^*\big(e_*\mathcal{A}_1\big) \xrightarrow{(\pi_1)^*e_*(\epsilon)} (\pi_1)^*e_*(\mathcal{O}_p).$$

Notice that $(\pi_1)^*(\mathcal{A})|_{e\times\mathcal{G}} = c^*e_*(\mathcal{O}_p)|_{e\times\mathcal{G}}$ and $(\pi_1)^*(\mathcal{A})|_{e\times\mathcal{G}} = (\pi_1)^*e_*(\mathcal{O}_p)|_{e\times\mathcal{G}}$. Also, $c^*e_*(\mathcal{O}_p) = (\pi_1)^*e_*(\mathcal{O}_p) = \mathbb{K}\boxtimes\mathbb{K} \cong \mathbb{K}$. Now the axiom demands $\alpha_1\big|_{e\times\mathcal{G}} = \alpha_2\big|_{e\times\mathcal{G}}$.

**Remarks.**

(i) Axiom (iii) in the definition above is superfluous and is only given for convenience. Indeed, it can be deduced from (ii) and injectivity of $\varphi$. The injectivity of $\varphi$ insures that $\epsilon\circ\varphi$ is also a counit. Then the uniqueness of a counit for a given comultiplication map implies that $\varphi$ preserves counit. In our setting this means, $c^*e_*(\epsilon)\circ\varphi = (\pi_1)^*e_*(\epsilon)$ on $e\times\mathcal{G}$.

(ii) For a commutative ring $S$, let $\mathbb{G}$ be the specialisation of $\mathcal{G}$ at $S$ i.e it is the algebraic group $\mathcal{G}(S)$. Then at the level of specialisation, a crossing becomes a set of algebra isomorphisms $\varphi = \{\varphi_{f,g} : A_f \to A_{gfg^{-1}}\}_{f,g\in\mathbb{G}}$. We can omit the first subscript and the crossings satisfies the following axioms:

- $\varphi$ is multiplicative ı.e., $\varphi_f \circ \varphi_g = \varphi_{fg}$ , for any $f, g \in \mathbb{G}$.
- $\varphi$ is compatible with $\Delta$ ı.e, $\Delta_{hfh^{-1},hgh^{-1}} \circ \varphi_h = (\varphi_h \otimes \varphi_h) \circ \Delta_{f,g}$ , for any $f, g, h \in \mathbb{G}$.
- $\varphi$ is compatible with $\epsilon$ ı.e, $\epsilon \circ \varphi_f = \epsilon$ , for any $f \in \mathbb{G}$.

All these three properties follows by specialising the axioms of a crossings at $S$ thus giving a *crossed Hopf* $\mathbb{G}$*-coalgebra* $A = \{A_g\}_{g\in\mathbb{G}}$ as defined by Virelizier [Vir02].

(iii) It follows that for any $g \in \mathbb{G}$, $\varphi_{g,e} : A_g \to A_g$ is an identity map using the multiplicativity of $\varphi$.

**Examples.** The two examples of Hopf $\mathcal{G}$-coalgebras discussed in section 4.2.1 are crossed. We give here the formula for the crossing for the two examples, [Vir05].

(i) $\mathbb{V}_n$: The family $\mathcal{A}_n$ of Hopf $GL_n(\mathbb{K})$-coalgebra as discussed in last section is crossed. The crossings are given by the following formulae:

$$\varphi_\alpha(g) = g, \quad \varphi_\alpha(x_i) = \sum_{k=1}^{n} \alpha_{k,i} x_k, \quad \varphi_\alpha(y_i) = \sum_{k=1}^{n} \alpha_{k,i} y_k$$

(ii) $\mathbb{U}_q$: The family $U_q^\pi(\mathfrak{g}) = \{U_q^\alpha(\mathfrak{g})\}_{\alpha\in\pi}$ is a crossed Hopf $\pi$-coalgebra. The crossings are given by:

$$\varphi_\alpha(K_i) = K_i, \quad \varphi_\alpha(E_i) = \alpha_i E_i, \quad \varphi_\alpha(F_i) = \alpha_i^{-1} F_i.$$

### 4.2.4 Quasitriangularity

Given a crossed Hopf $\mathcal{G}$-coalgebra $\mathcal{A}$, we now provide it with a quasi-triangular structure. First let us define a universal $\mathcal{R}$-matrix for $\mathcal{A}$ as the sheaf over $\mathcal{G} \times \mathcal{G}$ of invertible points in the global section ı.e

$$\mathcal{R} \in \Gamma(\mathcal{G}, \mathcal{A}) \otimes_{\mathbb{K}} \Gamma(\mathcal{G}, \mathcal{A}) \subseteq \Gamma(\mathcal{G} \times \mathcal{G}, \mathcal{A} \boxtimes \mathcal{A})$$

such that for any open $U \times V \subseteq \mathcal{G} \times \mathcal{G}$, $\mathcal{R}(U \times V)$ is an invertible element in the algebra $\Gamma(\mathcal{G}, \mathcal{A}) \otimes_{\mathbb{K}} \Gamma((\mathcal{G}, \mathcal{A})$. Thus at the level of specialisation, when $\mathbb{G} = \mathcal{G}(S)$ for some commutative ring $S$, a universal $R$-matrix is the collection of fibres:

$$\mathcal{R} = \{R_{f,g} \in \mathcal{A}_f \otimes \mathcal{A}_g : f, g \in \mathbb{G}\}$$

satisfying the following axioms:

(i) For any $g, h \in \mathbb{G}$,

$$\tau_{h,g}\Big((\varphi_{g^{-1}} \otimes 1_g)\Delta_{ghg^{-1},g}(x)\Big) = R_{g,h}\Delta_{g,h}(x)R_{g,h}^{-1} \tag{4.10}$$

where $\tau_{h,g}$ denotes the flip map $\mathcal{A}_h \otimes \mathcal{A}_g \to \mathcal{A}_g \otimes \mathcal{A}_h$.

(ii)

$$\big(1_f \otimes \Delta_{g,h}\big)R_{f,gh} = (R_{f,h})_{1g3}(R_{f,g})_{12h} \tag{4.11}$$

and,

$$\big(\Delta_{f,g} \otimes 1_h\big)R_{fg,h} = \Big((\varphi_g \otimes 1_h)(R_{g^{-1}fg,h})\Big)(R_{g,h})_{f23} \tag{4.12}$$

where for $\mathbb{K}$-algebras $X, Y$, and $r = \sum_j x_j \otimes y_j \in X \otimes Y$, we set

$$r_{12h} = r \otimes 1_h \in X \otimes Y \otimes A_h.$$

$$r_{f23} = 1_f \otimes r \in A_f \otimes X \otimes Y.$$

$$r_{1g3} = \sum_j x_j \otimes 1_g \otimes y_j \in X \otimes A_g \otimes Y.$$

(iii) the section $\mathcal{R}$ is invariant under the crossing i.e.,

$$\big(\varphi_f \otimes \varphi_f\big)(R_{g,h}) = R_{fgf^{-1},fhf^{-1}}. \tag{4.13}$$

A *quasitriangular Hopf $\mathcal{G}$-coalgebra* is a crossed Hopf $\mathcal{G}$-coalgebra $\mathcal{A} = \{\mathcal{A}, \Delta, \epsilon, S, \varphi\}$ endowed with a universal $\mathcal{R}$-matrix $\mathcal{R} \in \Gamma(\mathcal{G} \times \mathcal{G}, \mathcal{A} \boxtimes \mathcal{A})$. Note that the fiber at unity,

$R_{e,e}$ is the classical $R$-matrix for the Hopf algebra $A_e$ since $\varphi_e = Id$ by remark (iii) of last section. In case $\mathbb{G}$ is also an abelian group and the crossings $\varphi$ happen to be trivial (that is $\varphi_h|_{A_g} = 1_g$ for all $g, h \in \mathbb{G}$ ), one recovers the definition of a quasitriangular coloured Hopf algebra given by Ohtsuki for the discrete case, [Oht93] .

**Example.** The family $\mathcal{A}_n$ of crossed Hopf $GL_n(\mathbb{K})$-coalgebra as discussed in last section is quasitriangular. The $\mathbb{R}$-matrix is given, for $\alpha$, $\beta \in GL_n(\mathbb{K})$, by:

$$R_{\alpha,\beta} = \frac{1}{2} \sum_{S \subseteq [n]} x_S \otimes y_S + x_S \otimes g y_S + g x_S \otimes y_S - g x_S \otimes g y_S.$$

Here $[n] = \{1, \cdots n\}$, $x_\emptyset = 1$, $y_\emptyset = 1$ and for a nonempty subset $S$ of $n$ we let $x_S = x_{i_1} \cdots x_{i_S}$ and $y_S = y_{i_1} \cdots y_{i_S}$ where $i_1 < \cdots < i_S$ are the elements of $S$.

Note that the family $U_q^\pi(\mathfrak{g}) = \{U_q^\alpha(\mathfrak{g})\}_{\alpha \in \pi}$ which is a crossed Hopf $\pi$-coalgebra is not quasitriangular in general. Indeed, it is topologically quasitriangular so that the universal $\mathcal{R}$-matrix will lie in the completion of the external tensor product of the sheaf $\mathcal{A}$ with itself. Note that usual $U_q(\mathfrak{g})$ is not quasitriangular either. Since $R$ lies in some completeion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ rather than in $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ itself. Drinfeld gives an expression for $R$ as a power series in tensor products of the generators of $U_q(\mathfrak{g})$, [Dri87] .

## 4.3 Affine case

In this section, we shall assume that the underlying group scheme is affine. Let $\mathcal{G} = \operatorname{Spec}(H)$ for a commutative $\mathbb{K}$-Hopf algebra $H$. This helps us to elaborate the dual of a Hopf $\mathcal{G}$-coalgebra, crossings for a Hopf $\mathcal{G}$-coalgebra and thus formulate the quantum double explicitly.

### 4.3.1 Structure of a Hopf $\mathcal{G}$-coalgebra

A quasicoherent sheaf $\mathcal{A}$ of algebras over the affine group scheme $\mathcal{G}$ is simply a quasicoherent sheaf $\widetilde{A}$ where $A$ is an $H$-algebra with the action of $H$ on $A$ given via $^- : H \to A$. At any topological point $f \in \mathcal{G}$, the stalk of $\widetilde{A}$ at $f$ is $A_{(f)} = H_{(f)} \otimes_H A$ where $H_{(f)}$ is the localisation of $H$ at the point $f$ (which is essentially an ideal). Let $\mathbb{S}$ be a commutative $\mathbb{K}$-algebra. Considering $\mathcal{G}$ as a functor, the fiber of $\widetilde{A}$ over any algebraic point $f : H \to \mathbb{S}$

is $\mathbb{S} \otimes_H A$. Note that $\mathbb{S}$ is an $H$-module via the map $f$. From here onwards we choose the notation $\mathbb{S} \otimes_f A$ to refer to the fiber of $\widetilde{A}$ at the algebraic point $f : H \to \mathbb{S}$. Now given that the sheaf $\widetilde{A}$ is a Hopf $\mathcal{G}$-coalgebra implies that it is equipped with following structure maps which can be visualised explicitly at the level of fibres :

$$\Delta_{f,g} : \mathbb{S} \otimes_{\mu(f,g)} A \to \mathbb{S} \otimes_f A \otimes_{\mathbb{S}} \mathbb{S} \otimes_g A$$

$$\epsilon : \mathbb{S} \otimes_1 A \to \mathbb{S}$$

$$S_f : \mathbb{S} \otimes_f A \to \mathbb{S} \otimes_{f^{-1}} A$$

where $f, f^{-1}, g, 1 : H \to \mathbb{S}$ are all algebraic points in $\mathcal{G}(\mathbb{S})$ with $f^{-1} : H \to \mathbb{S}$ is such that $f(x_1)f^{-1}(x_2) = \epsilon(x).1_{\mathbb{S}}$ and $1 = \eta_{\mathbb{S}} \circ \epsilon_H$.

Globally, the structures $\Delta$ and $\epsilon$ are morphisms of algebras such that they satisfy respectively the axiom of coassociativity and the axiom of counity. Note that at global level, $\Gamma(\mathcal{G}, \widetilde{A}) = A$, and by Theorem (4.2.2) $A$ is essentially a Hopf algebra over $\mathbb{K}$.

### 4.3.2 Crossings

The Hopf $\mathcal{G}$-coalgebra sheaf $\widetilde{A}$ is crossed implies that it is equipped with following isomorphisms of algebras (called crossings), which at the level of fibres is given as :

$$\varphi_g : \mathbb{S} \otimes_f A \to \mathbb{S} \otimes_{c(f,g)} A.$$

Note that $c(f, g) = gfg^{-1} : H \to \mathbb{S}$ is again an algebraic point of $\mathcal{G}$ defined as $x \mapsto g(x_1)f(x_2)g^{-1}(x_3)$, where $g^{-1}(y) = g(Sy)$. These crossings satisfy the following two axioms:

(i) $\varphi$ preserves comultiplication of $A$. This can be expressed at fiber-level by the following diagram:

$$\begin{tikzcd}
\mathbb{S} \otimes_{\mu(f,g)} A \arrow[r, "\varphi_h"] \arrow[d, "\Delta"'] & \mathbb{S}\otimes_{c(fg,h)} A \arrow[d, "\Delta"] \\
(\mathbb{S} \otimes_f A) \otimes_{\mathbb{S}} (\mathbb{S} \otimes_g A) \arrow[r, "\varphi_h \otimes \varphi_h"] & (\mathbb{S} \otimes_{c(f,h)} A) \otimes_{\mathbb{S}} (\mathbb{S} \otimes_{c(g,h)} A).
\end{tikzcd}$$

(ii) $\varphi$ preserves counit of $A$. Explicitly,

$$\begin{tikzcd}
\mathbb{S} \otimes_1 A \arrow[r, "\varphi_g"] \arrow[dr, "\mathbb{S}\otimes\epsilon"'] & \mathbb{S} \otimes_{c(1,g)} A \arrow[d, "\mathbb{S}\otimes\epsilon"] \\
 & \mathbb{S}.
\end{tikzcd}$$

Note that $\mathbb{S} \otimes_{c(1,g)} A = \mathbb{S} \otimes_1 A$ as $c(1, g) = g1g^{-1} = 1$.

**Remarks.**

(a) Axiom (ii) in the definition above is superfluous and is only given for convenience. Indeed, it can be deduced from (i) and injectivity of $\varphi$. The injectivity of $\varphi$ ensures that $\epsilon \circ \varphi$ is also a counit. Then the uniqueness of a counit for a given comultiplication map implies that $\varphi$ preserves counit.

(b) Note that the crossing preserves the antipode, i.e. for any $g, h \in \mathbb{G}$, $\varphi_g S_h = S_{ghg^{-1}} \varphi_g$.

(c) At the global level, crossing is essentially a coaction of $H$ on $A$ given by

$$\varphi : A \to A \otimes H : a \mapsto a_{[0]} \otimes a_{[1]},$$

such that for any $x \in H$, $\varphi : x \mapsto x_2 \otimes (Sx_1)x_3$ is the right adjoint coaction of $H$ on itself. The axioms of the crossing can be interpreted at the global level as follows:

- Multiplicativity of $\varphi$ is the coassociativity of the coaction which implies:

$$a_{[0][0]} \otimes \overline{a}_{[0][1]} \otimes \overline{a}_{[1]} = a_{[0]} \otimes \overline{a}_{[1]1} \otimes \overline{a}_{[1]2} \tag{4.14}$$

where $^- : H \hookrightarrow A$ is the inclusion map of $\mathbb{K}$-algebras.

- Crossing preserves multiplication gives the following formula:

$$(ab)_{[0]} \otimes \overline{(ab)}_{[1]} = a_{[0]}b_{[0]} \otimes \overline{a}_{[1]}\overline{b}_{[1]} \tag{4.15}$$

- Crossing preserves comultiplication gives the following formula:

$$a_{[0]1} \otimes a_{[0]2} \otimes \overline{a}_{[1]} = a_{1[0]} \otimes a_{2[0]} \otimes \overline{a}_{1[1]}\overline{a}_{2[1]} \tag{4.16}$$

Note that applying comultiplication twice gives:

$$a_{[0]1} \otimes a_{[0]2} \otimes a_{[0]3} \otimes \overline{a}_{[1]} = a_{1[0]} \otimes a_{2[0]} \otimes a_{3[0]} \otimes \overline{a}_{1[1]}\overline{a}_{2[1]}\overline{a}_{3[1]} \tag{4.17}$$

- Crossing preserves counit gives the following formula:

$$(a)_{[0]} \otimes \overline{(a)}_{[1]} = \epsilon(a)1. \tag{4.18}$$

- Finally crossing preserves antipode implies that:

$$(Sa)_{[0]} \otimes \overline{(Sa)}_{[1]} = Sa_{[0]} \otimes \overline{a}_{[1]}. \tag{4.19}$$

We shall be using these formulae for defining a quantum double. The fibre at $x \in \mathcal{G}(\mathbb{S})$ gives, for any $y \in \mathcal{G}(\mathbb{S})$ :

$$\varphi_x : \mathbb{S} \otimes_y A \to \mathbb{S} \otimes_{xyx^{-1}} A$$

such that $s \otimes a \mapsto x(a_{[1]})s \otimes a_{[0]}$. It satisfies usual crossing axioms , given by Zunino/Turaev (cf 4.3.3). Note that for a point $P : H \to H$ in $\mathcal{G}(H)$, crossing $\varphi$ will be an endomorphism of $A$. In this case one can think of $\varphi_e$ as a specialisation of $\varphi_I$ where $I : H \to H$ is the identity map and $e$ is the identity of the group $\mathcal{G}(H)$.

Consider the dual cosheaf $\mathcal{A}^*$ of the sheaf $\mathcal{A}$ (cf. Section (2.4.3)). We provide $\mathcal{A}^*$ with an equivariant structure using the crossing of $\mathcal{A}$. This is simply the dual coaction of the crossing of $\mathcal{A}$ given as follows:

$$\varphi : A \to A \otimes H$$

$$\psi : A^* \to A^* \otimes H.$$

Fibres are:

$$\psi_{\alpha,\beta} : \mathcal{A}^*_{\beta\alpha\beta^{-1}} \to \mathcal{A}^*_{\alpha}$$

We will have, $\psi_{\alpha,\beta}\big(f_{\beta\alpha\beta^{-1}}\big)(a_\alpha) = f_{\beta\alpha\beta^{-1}}\big(\varphi_{\alpha,\beta}(a_\alpha)\big)$, where $f_{\beta\alpha\beta^{-1}} \in \mathcal{A}^*_{\beta\alpha\beta^{-1}}$ and $a_\alpha \in \mathcal{A}_\alpha$.

Note that the collection of fibres $\{A_f\}_{f\in\mathbb{G}}$ forms a crossed Hopf $\mathbb{G}$-coalgebra over $\mathbb{S}$, where $\mathbb{G} = \mathcal{G}(\mathbb{S})$. By the abuse of notation, let us also denote this collection by $A$. Given a Hopf group coalgebra $A$, a Hopf algebra $A_{\text{tot}}$ over $\mathbb{S}$ is defined as follows. As a coalgebra,

$$A_{\text{tot}} = \bigoplus_{f\in\mathbb{G}} A_f$$

such that $A_f \otimes A_g \subset A_{fg}$ for any $f, g \in \mathbb{G}$. The rest of the structure maps are just the sums of the individual structure maps, i.e. $\varphi_{\text{tot,f}} = \sum_{g\in\mathbb{G}} \varphi_f : \bigoplus_{g\in\mathbb{G}} A_g \mapsto \bigoplus_{g\in\mathbb{G}} A_g$ and $S_{\text{tot}} = \sum_{f\in\mathbb{G}} S_f$, such that $S_{\text{tot}}(A_f) = A_{f^{-1}}$ and $1 \in A_1$. Further, Zunino describes the Hopf $\mathbb{G}$-coalgebra $\overline{A}$, called the *mirror* of $A$ defined as follows:

- For any $f \in \mathbb{G}$, we set $A_f = A_{f^{-1}}$.
- For any $f, g \in \mathbb{G}$, the component $\overline{\Delta}_{f,g}$ of the comultiplication $\overline{\Delta}$ of $\overline{A}$ is given by

$$\overline{\Delta}_{f,g}(a) = \Big((\varphi_g \otimes A_{g^1}) \circ \Delta_{g^{-1}fg,g^{-1}}\Big)(a) \in A_{f^{-1}} \otimes A_{g^{-1}} = \overline{A_f} \otimes \overline{A_g},$$

  for any $a \in A_{g^1 f^{-1}} = \overline{A}_{fg}$.

- The counit of $\overline{A}$ is given by $\epsilon : A_1 \to \mathbb{S}$.
- For any $f, g \in \mathbb{G}$, the component of the antipode $\overline{s}$ of $A$ is given by $\overline{s}_f = \varphi \circ s_{f^{-1}}$.
- Finally, we set the crossing as $\overline{\varphi_f} = \varphi_f$.

These two constructions namely, $\overline{A}$ and $A_{\text{tot}}$ help in defining Zunino's quantum double which we discuss in the Section 4.3.4.

### 4.3.3 Quasitriangularity

Following on from 4.2.4, at affine level quasitriangularity is equivalent to having an invertible element $R \in A \otimes A$ such that it satisfies the following axioms:

(i) $\tau(\varphi \otimes 1)\Delta = R\Delta R^{-1}$

(ii) $(1 \otimes \Delta)R = R_{13}R_{12}$

(iii) $(\Delta \otimes 1)R = \big[(\varphi \otimes 1)R_{13}\big]R_{23}$

(iv) $(\varphi \otimes \varphi)R = R.$

Note that if crossing is trivial we recover the classical definition of a universal $R$-matrix for a Hopf algebra.

## 4.4 Quantum double

Before we express the definition of the Drinfeld quantum double in our new setting of algebraic groups at the level of fibers, we state the critical difference of the quantum double in our new setting from the one of Zunino, [Zun04a]. He requires his Hopf group coalgebra, $H = \{H_\alpha\}_{\alpha\in\pi}$ (what he calls as a Turaev-coalgebra) to be of finite type. This requires every component of the collection $\{H_\alpha\}$ to be a finite dimensional algebra over a field $\mathbb{K}$. In our case we will perform the construction over a ring without any finite dimensionality requirement.

### 4.4.1 Zunino's Quantum double

Zunino introduces a $T$-coalgebra, $H^{*tot}$, the inner dual of $H$. It is the one we discussed in Section (3.1). The components of $H^{*tot}$ are all isomorphic as algebras and, as a vector space, $H^{*tot} = \bigoplus_{\beta\in\pi} H^*_\beta$ for any $\alpha \in \pi$. And then each component of the quantum double defined by him is, as a vector space, $H_{\alpha^{-1}} \otimes \Big(\bigoplus_{\beta\in\pi} H^*_\beta\Big)$.

Thus, for any $\alpha \in \pi$, the $\alpha^{\text{th}}$ component of quantum double $D(H)$, denoted by $D_\alpha(H)$

is a vector space, given as

$$D(H)_\alpha = H_{\alpha^{-1}} \otimes \big( \oplus_{\beta \in \pi} H^*_\beta \big).$$

For every $\alpha \in \pi$, $D_\alpha(H) = D(H)_\alpha$ is an algebra under the multiplication obtained by setting, for any $a_{\alpha^{-1}}$, $b_{\alpha^{-1}} \in H_{\alpha^{-1}}$, $f_\gamma \in H^*_\gamma$ and $g_\delta \in H^*_\delta$ with $\alpha$, $\gamma$ and $\delta \in \pi$,

$$(a_{\alpha^{-1}} \otimes f_\gamma).(b_{\alpha^{-1}} \otimes g_\delta) = a^{''}_{\alpha^{-1}} b \otimes f_\gamma g_\delta \Big( \overline{S}_\delta (a^{'''}_{\delta^{-1}}) \_ \varphi_\alpha (a^{'}_{\alpha^{-1}\delta\alpha}) \Big), \tag{4.20}$$

where comultiplication when applied twice to $a_{\alpha^{-1}}$ yields $a^{'}_{\alpha^{-1}\delta\alpha}$, $a^{''}_{\alpha^{-1}}$, and $a^{'''}_{\delta^{-1}}$, which are respectively in $H_{\alpha^{-1}\delta\alpha}$, $H_{\alpha^{-1}}$ and $H_{\delta^{-1}}$. For short hand, let us denote this multiplication as :

$$a_{\alpha^{-1}} \otimes g_\delta \mapsto a^{''}_{\alpha^{-1}} \otimes g_\delta \Big( \overline{S}_\delta (a^{'''}_{\delta^{-1}}) \_ \varphi_\alpha (a^{'}_{\alpha^{-1}\delta\alpha}) \Big) \tag{4.21}$$

Note that here $g_\delta \Big( \bar{S}_\delta (a^{'''}_{\delta^{-1}}) \_ \varphi_\alpha (a^{'}_{\alpha^{-1}\delta\alpha}) \Big)$ means the map $x \mapsto g_\delta \Big( \bar{S}_\delta (a^{'''}_{\delta^{-1}}) x \varphi_\alpha (a^{'}_{\alpha^{-1}\delta\alpha}) \Big)$ for $x \in H_\delta$. There can be yet another multiplication formula for quantum double in discrete case which arises using the comultiplication of the dual. This can be written in the short hand notation as follows:

$$a_{\alpha^{-1}} \otimes g_\delta \mapsto \Big[ \psi_{\alpha^{-1}} (g^{'''}_\delta)(a^{'}_{\alpha^{-1}\delta\alpha}) \overline{S}^*_{\delta^{-1}} (g^{'}_\delta)(a^{'''}_{\delta^{-1}}) \Big] a^{''}_{\alpha^{-1}} \otimes g^{''}_\delta \tag{4.22}$$

where comultiplication when applied twice to $g_\delta$ yields $g^{'}_\delta$, $g^{''}_\delta$, and $g^{'''}_\delta$, all being elements in $H^*_\delta$.

**Proposition 4.4.1** *The two multiplication formulae described above for a discrete quantum double are equivalent.*

PROOF:

$$\begin{aligned}
\text{Right hand side of (4.22)} &= a''_{\alpha^{-1}} \otimes \Big[\psi_{\alpha^{-1}}(g'''_\delta)(a'_{\alpha^{-1}\delta\alpha})\overline{S}^*_{\delta^{-1}}(g'_\delta)(a'''_{\delta^{-1}})\Big] g''_\delta \\
&= a''_{\alpha^{-1}} \otimes \Big[g'''_\delta\left(\varphi_\alpha(a'_{\alpha^{-1}\delta\alpha})\right) g'_\delta\left(\overline{S_\delta}(a'''_{\delta^{-1}})\right)\Big] g''_\delta \\
&= a''_{\alpha^{-1}} \otimes g_\delta\Big[\overline{S}_\delta(a'''_{\delta^{-1}})_\varphi_\alpha(a'_{\alpha^{-1}\delta\alpha})\Big] \\
&= \text{Right hand side of (4.21)}.
\end{aligned}$$

□

Let us now discuss and realise the quantum double structure given by Zunino. Here is the interpretation of his construction in the affine case. We have seen that the collection of fibres of a Hopf $\mathcal{G}$-coalgebra $\mathcal{A}$ at a specialisation $\mathbb{S}$ forms a Hopf $\mathbb{G}$-coalgebra, say $\mathcal{A}_\mathbb{S} = \{A_f\}_{f\in\mathbb{G}}$ with values in $\mathbb{S}$. Then by Zunino's construction $\overline{\mathcal{A}_\mathbb{S}}$ and $(\mathcal{A}^\circ_\mathbb{S})^{\mathrm{tot,cop}}$ also forms a Hopf $\mathbb{G}$-coalgebra over $\mathbb{S}$ (Section 4.3.2). Here the superscript cop means that comultiplication and antipode are opposite i.e. :

$$(\Delta^\circ)^{\mathrm{cop}}(\alpha)(a\otimes b) = \alpha(ba)$$

$$(S^\circ)^{\mathrm{cop}} = \left(S^\circ\right)^{-1}$$

where $a, b \in A$, $\alpha \in A^\circ$ and $\Delta^\circ$, $S^\circ$ are respectively the comultiplication and antipode of $A^\circ$. Recall that $A^\circ$ has elements vanishing over coprojective ideal. In particular we have $(S^\circ)^{\mathrm{cop}}(\alpha) : a \mapsto \alpha\big(S^{-1}(a)\big)$. Zunino's quantum double construction, [Zun04a], gives a Hopf $\mathbb{G}$-coalgebra $D_Z(\mathcal{A}_\mathbb{S}) = \overline{A}\otimes(A^\circ)^{\mathrm{tot,cop}}$ such that its component at some point $f \in \mathbb{G}$ is given by:

$$D_Z(\mathcal{A}_\mathbb{S})_f = A_{f^{-1}} \otimes \big[\bigoplus_{g\in\mathbb{G}} A^\circ_g\big]$$

Observe that for every $f \in \mathbb{G}$, $A_f = \mathbb{S}\otimes_f A$ is an $\mathbb{S}$-algebra. Moreover, $D_Z(\mathcal{A}_\mathbb{S})_f$ is an $\mathbb{S}$-algebra under the multiplication given by (4.20). Let us say it explicitly again,

$$(a_{f^{-1}}\otimes\alpha_g).(b_{f^{-1}}\otimes\beta_h) = a''_{f^{-1}}b_{f^{-1}}\otimes\alpha_g\beta_h\Big(\overline{s}_h(a'''_{h^{-1}})\_\varphi_f(a'_{f^{-1}hf})\Big),$$

where $\overline{s}$ is antipode and $\varphi$ is crossing of $\overline{\mathcal{A}_\mathbb{S}}$. The unit of $D_Z(\mathcal{A}_\mathbb{S})_f$ is given by $1_{f^{-1}}\otimes\epsilon$.

With these structures, every component of Zunino's double is an $\mathbb{S}$-algebra with action of $\mathbb{S}$ from left. Now if $A$ is flat over $H$, then $A_f$ becomes flat as an $\mathbb{S}$-algebra. Then, we have canonical embeddings $A_{f^{-1}}, \bigoplus A^{\circ}_g \hookrightarrow D_Z(\mathcal{A}_{\mathbb{S}})_f$ which are $\mathbb{S}$-algebra morphisms.

The comultiplication is given by :

$$\Delta_{f,g}(a \otimes \alpha) = \left(\varphi_g(a^{'}_{g^{-1}f^{-1}g}) \otimes \alpha^{''}\right) \otimes \left(a^{''}_{g^{-1}} \otimes \alpha^{'}\right). \tag{4.23}$$

The counit is obtained by setting, for any $\alpha \in A^{\circ}_g$,

$$\epsilon(a \otimes \alpha) = \epsilon(a)\alpha(1_g), \tag{4.24}$$

For any $f \in \mathbb{G}$, the component of the antipode of $D_Z(\mathcal{A}_{\mathbb{S}})$ at $g$ is given by

$$S_g(a \otimes \alpha) = \Big(\varphi_g S_{g^{-1}}(a) \otimes 1\Big)\Big(1_g \otimes s^{\circ}(\alpha)\Big), \tag{4.25}$$

where $a \in \overline{A}_g = A_{g^{-1}}$, $\overline{S}$ is the antipode of $\overline{A}_{g^{-1}}$, $s^{\circ}$ is the antipode $A^{\circ}$ and 1 in the second factor essentially means $\epsilon$. Finally, the crossing is defined as

$$\varphi_g(a \otimes \alpha) = \varphi_g(a) \otimes \varphi^{*}_{g^{-1}}(\alpha), \tag{4.26}$$

Then we have the following result of Zunino:

**Theorem 4.4.2** *[Zun04a] Given that $A$ is flat over $H$, $D_Z(\mathcal{A}_{\mathbb{S}})$ forms a crossed Hopf $\mathbb{G}$-coalgebra.*

Proof: Zunino proved this theorem when $\mathbb{S}$ is a field and each $A_g$ is a finite dimensional vector space over $\mathbb{K}$. Our conditions that $\mathbb{S}$ is a commutative $\mathbb{K}$-algebra, each $A_g$ is flat and taking finite dual instead of $A^*$ makes all the maps well defined. All algebraic identities are checked identically as in Zunino's case. □

### 4.4.2 Quantum double in affine case

In our setting, we are working with finite dual cosections of the dual cosheaf $\mathcal{A}^{\circ}$. Thus finite dimensionality is not required explicitly. As a result, our quantum double will be a quotient of the quantum double given by Zunino's construction. We discuss below the definition of Drinfeld quantum double in the affine setting. We define Drinfeld double of a crossed Hopf $\mathcal{G}$-coalgebra $\mathcal{A}$ as the sheaf $\widetilde{D(A)}$ such that:

$$D(A) := A \otimes_{\mathbb{K}} A^{\circ}$$

where $\mathcal{G} = \mathrm{Spec}\,(H)$ for a commutative $\mathbb{K}$-Hopf algebra $H$ and $A$ is an $H$-algebra with the action of $H$ on $A$ such that $^{-} : H \hookrightarrow A$ as a central Hopf $\mathbb{K}$-subalgebra and the crossing of $A$ is such that $\varphi : a \mapsto a_{[0]} \otimes a_{[1]}$. Thus one can observe that $D(A)$ also becomes an $H$-module via the action of $H$ on $A$ from left. We provide an algebra structure to $D(A)$ via the following multiplication formula:

$$(a \otimes \alpha)(b \otimes \beta) = a_2 b S\overline{a}_{1[1]} \otimes \alpha\beta\big(S^{-1}a_{3\_}a_{1[0]}\big) \tag{4.27}$$

where $a, b \in A$, $a_{1[0]} \in A$, $\overline{a}_{1[1]} \in H$ and $\alpha, \beta \in A^{\circ}$.

**Lemma 4.4.3** $D(A)$ *becomes an associative* $\mathbb{K}$-*algebra with the multiplication defined above.*

Proof: The multiplication defined in (4.27) is associative if and only if, for any $a, b, c \in A$, and $\alpha, \beta, \gamma \in A^{\circ}$,

$$\big[(a \otimes \alpha)(b \otimes \beta)\big](c \otimes \gamma) = (a \otimes \alpha)\big[(b \otimes \beta)(c \otimes \gamma)\big]. \tag{4.28}$$

By computing the left-hand side of (4.28), we obtain $\big[(a\otimes\alpha)(b\otimes\beta)\big](c\otimes\gamma)$

$$\begin{aligned}
&= \Big[a_2bS\overline{a}_{1[1]}\otimes\alpha.\beta\big(S^{-1}a_{3_}a_{1[0]}\big)\Big](c\otimes\gamma)\\
&= \big(a_2bS\overline{a}_{1[1]}\big)_2 cS\overline{\big(a_2bS\overline{a}_{1[1]}\big)}_{1[1]}\otimes\alpha\beta\big(S^{-1}a_{3_}a_{1[0]}\big)\gamma\Big(S^{-1}\big(a_2bS\overline{a}_{1[1]}\big)_{3_}\big(a_2bS\overline{a}_{1[1]}\big)_{1[0]}\Big)\\
&= \big(a_3b_2S\overline{a}_{1[1]2}cS\overline{a}_{2[1]}S\overline{b}_{1[1]}S\big(S\overline{a}_{1[1]3}\big)_{[1]}\otimes\alpha\beta\big(S^{-1}a_{5_}a_{1[0]}\big)\\
&\quad\gamma\Big(\overline{a}_{1[1]1}S^{-1}b_3S^{-1}a_{4_}a_{2[0]}b_{1[0]}\big(S\overline{a}_{1[1]3}\big)_{[0]}\Big)\\
&= a_3b_2S\overline{a}_{1[1]2}cS\overline{a}_{2[1]}S\overline{b}_{1[1]}S\big(\overline{a}_{1[1]5}S\overline{a}_{1[1]3}\big)\otimes\alpha\beta\big(S^{-1}a_{5_}a_{1[0]}\big)\\
&\quad\gamma\Big(\overline{a}_{1[1]1}S^{-1}b_3S^{-1}a_{4_}a_{2[0]}b_{1[0]}\big(S\overline{a}_{1[1]4}\big)\Big)\\
&= a_3b_2S\overline{a}_{1[1]2}cS\overline{a}_{2[1]}S\overline{b}_{1[1]}S\overline{a}_{1[1]5}\overline{a}_{1[1]3}\otimes\alpha\beta\big(S^{-1}a_{5_}a_{1[0]}\big)\\
&\quad\gamma\Big(\overline{a}_{1[1]1}S^{-1}b_3S^{-1}a_{4_}a_{2[0]}b_{1[0]}S\overline{a}_{1[1]4}\Big)\\
&= a_3b_2cS\overline{a}_{2[1]}S\overline{b}_{1[1]}S\overline{a}_{1[1]}\otimes\alpha\beta\big(S^{-1}a_{5_}a_{1[0]}\big)\gamma\Big(S^{-1}b_3S^{-1}a_{4_}a_{2[0]}b_{1[0]}\Big)
\end{aligned}$$

On the right-hand side of (4.28) we have, $(a\otimes\alpha)\big[(b\otimes\beta)(c\otimes\gamma)\big]$

$$\begin{aligned}
&= (a\otimes\alpha)\Big[b_2cS\overline{b}_{1[1]}\otimes\beta.\gamma\big(S^{-1}b_{3_}b_{1[0]}\big)\Big]\\
&= a_2b_2c\overline{b}_{1[1]}S\overline{a}_{1[1]}\otimes\alpha\Big[\beta.\gamma\big(S^{-1}b_{3_}b_{1[0]}\big)\Big]\big(S^{-1}a_{3_}a_{1[0]}\big)\\
&= a_2b_2c\overline{b}_{1[1]}S\overline{a}_{1[1]}\otimes\alpha\beta\big(S^{-1}a_{4_}a_{1[0]_1}\big)\gamma\Big(S^{-1}b_3S^{-1}a_{3_}a_{1[0]2}b_{1[0]}\Big)\\
&= a_2b_2c\overline{b}_{1[1]}S\big(\overline{a}_{1[1]}\overline{a}_{2[1]}\big)\otimes\alpha\beta\big(S^{-1}a_{4_}a_{1[0]}\big)\gamma\Big(S^{-1}b_3S^{-1}a_{3_}a_{2[0]}b_{1[0]}\Big)\\
&= a_3b_2c\overline{b}_{1[1]}S\overline{a}_{1[1]}S\overline{a}_{2[1]}\otimes\alpha\beta\big(S^{-1}a_{5_}a_{1[0]}\big)\gamma\Big(S^{-1}b_3S^{-1}a_{4_}a_{2[0]}b_{1[0]}\Big)
\end{aligned}$$

We have used the following identity :

$$\overline{a}_{2[1]}\ \ \overline{a}_{1[1]}\otimes a_{1[0]}\otimes a_{2[0]}=\overline{a}_{1[1]}\otimes a_{1[0]1}\otimes a_{1[0]2},$$

which is the axiom of crossing being preserved under comultiplication (4.16). □

Thus $D(A)$ so defined is a $\mathbb{K}$-algebra. We now provide it with a comultiplication:

$$\Delta(a\otimes\alpha)=\big(a_{1[0]}\otimes\alpha_2\big)\otimes\big(S\overline{a}_{1[1]}a_2\otimes\alpha_1\big). \tag{4.29}$$

where $a \in A$, $\alpha \in A^\circ$ and comultiplication of $A^\circ$ is such that $\alpha \mapsto \alpha_1 \otimes \alpha_2$. Counit is given by the following formula:

$$\epsilon : A \otimes A^\circ \longrightarrow \mathbb{K} \ : \epsilon(a \otimes \alpha) = \epsilon(a)\alpha(1). \tag{4.30}$$

**Lemma 4.4.4** *$D(A)$ becomes a coassociative $\mathbb{K}$-coalgebra with the comultiplication and counit given respectively by (4.29) and (4.30).*

Proof: The comultiplication defined in (4.29) is coassociative if and only if, for any $a, b, c \in A$, and $\alpha, \beta, \gamma \in A^\circ$, following holds:

$$\big(\Delta \otimes 1\big)\Delta(a \otimes \alpha) = \big(1 \otimes \Delta\big)\Delta(a \otimes \alpha). \tag{4.31}$$

Let us first compute the left-hand side of (4.31). Since the second factor does not involve any calculations, we evaluate (4.31) only for first factor: $\big(\Delta \otimes 1\big)\Delta(a)$

$$\begin{aligned}
&= \big(\Delta \otimes 1\big)\big(a_{1[0]} \otimes S\overline{a}_{1[1]}a_2\big)\\
&= \Delta\big(a_{1[0]}\big) \otimes S\overline{a}_{1[1]}a_2\\
&= \big(a_{1[0]}\big)_{1[0]} \otimes S\big(a_{1[0]}\big)_{1[1]}\, a_{1[0]2} \otimes S\overline{a}_{1[1]}a_2\\
&= a_{1[0]1[0]} \otimes Sa_{1[0]1[1]}\, a_{1[0]2} \otimes S\overline{a}_{1[1]}a_2\\
&= a_{11[0][0]} \otimes Sa_{11[0][1]}\, a_{12[0]} \otimes S\big(\overline{a}_{11[1]}\overline{a}_{12[1]}\big)\, a_2\\
&= a_{1[0][0]} \otimes S\overline{a}_{1[0][1]}\, a_{2[0]} \otimes S\big(\overline{a}_{1[1]}\overline{a}_{2[1]}\big)a_3\\
&= a_{1[0]} \otimes S\overline{a}_{1[1]1}\, a_{2[0]} \otimes S\big(\overline{a}_{1[1]2}\overline{a}_{2[1]}\big)a_3\\
&= a_{1[0]} \otimes S\overline{a}_{1[1]1}\, a_{2[0]} \otimes S\overline{a}_{2[1]}S\overline{a}_{1[1]2}a_3.
\end{aligned}$$

Computing the right-hand side of (4.31) for first factor only, we have $\big(1 \otimes \Delta\big)\Delta(a)$ :

$$
\begin{aligned}
&= \big(1 \otimes \Delta\big)\big(a_{1[0]} \otimes S\overline{a}_{1[1]}a_2\big) \\
&= a_{1[0]} \otimes \Delta\big(S\overline{a}_{1[1]}a_2\big) \\
&= a_{1[0]} \otimes \big(S\overline{a}_{1[1]}a_2\big)_{1[0]} \otimes S\overline{\Big(S\overline{a}_{1[1]}a_2\Big)}_{1[1]}\ \big(S\overline{a}_{1[1]}a_2\big)_2 \\
&= a_{1[0]} \otimes \big(S\overline{a}_{1[1]}\big)_{1[0]}a_{21[0]} \otimes S\Big(\overline{\big(S\overline{a}_{1[1]}\big)}_{1[1]}\overline{a_{21[1]}}\Big)\ \big(S\overline{a}_{1[1]}\big)_2\, a_{22} \\
&= a_{1[0]} \otimes S\overline{a}_{1[1]2[0]}a_{2[0]} \otimes S\Big(\overline{a}_{1[1]2[1]}\overline{a}_{2[1]}\Big)\ S\overline{a}_{1[1]1}\, a_3 \\
&= a_{1[0]} \otimes S\overline{a}_{1[1]3}a_{2[0]} \otimes S\Big(\overline{a}_{1[1]4}S\overline{a}_{1[1]2}\overline{a}_{2[1]}\Big)\ S\overline{a}_{1[1]1}\, a_3 \\
&= a_{1[0]} \otimes S\overline{a}_{1[1]3}a_{2[0]} \otimes S\overline{a}_{2[1]}\overline{a}_{1[1]2}S\overline{a}_{1[1]4}\ S\overline{a}_{1[1]1}\, a_3 \\
&= a_{1[0]} \otimes S\overline{a}_{1[1]1}a_{2[0]} \otimes S\overline{a}_{2[1]}S\overline{a}_{1[1]2}\, a_3.
\end{aligned}
$$

Note that we have used axioms of crossing (4.14)-(4.19) in the above calculations and that $H$ is commutative.

We now show the multiplicativity of the counit which requires : $\epsilon\Big((a \otimes \alpha)(b \otimes \beta)\Big) = \epsilon(a \otimes \alpha)\epsilon(b \otimes \beta)$. The right side of this equation equals $\epsilon(a)\alpha(1)\epsilon(b)\beta(1)$ whereas the left side which is : $\epsilon\Big((a \otimes \alpha)(b \otimes \beta)\Big)$

$$
\begin{aligned}
&= \epsilon\Big(a_2bS\overline{a}_{1[1]} \otimes \alpha\beta\big(S^{-1}a_{3\_}a_{1[0]}\big)\Big) \\
&= \epsilon\Big(a_2bS\overline{a}_{1[1]}\Big)\alpha(1)\beta\big(S^{-1}a_3a_{1[0]}\big) \\
&= \epsilon(a_2)\epsilon(b)\epsilon(S\overline{a}_{1[1]})\alpha(1)\beta\big(S^{-1}a_3a_{1[0]}\big) \\
&= \epsilon(b)\epsilon(S\overline{a}_{1[1]})\alpha(1)\beta\big(S^{-1}a_2a_{1[0]}\big) \\
&= \epsilon(b)\alpha(1)\beta\big(S^{-1}a_2a_1\big) \\
&= \epsilon(b)\alpha(1)\beta\big(\epsilon(a)1\big) \\
&= \epsilon(a)\alpha(1)\epsilon(b)\beta(1)
\end{aligned}
$$

which completes the proof. □

We now provide $D(A)$ with a crossing:

$$
\varphi(a \otimes \alpha) = \big(a_{[0]} \otimes \alpha_{[0]}\big) \otimes \overline{a}_{[1]}S\overline{\alpha}_{[1]}. \tag{4.32}
$$

**Lemma 4.4.5** *The crossing given by equation (4.32) is an algebra morphism.*

PROOF: The crossing defined in (4.32) will be an algebra morphism if and only if for any $a, b \in A$, and $\alpha, \beta \in A^\circ$, following holds:

$$\varphi(a \otimes \alpha)\varphi(b \otimes \beta) = \varphi\Big((a \otimes \alpha)(b \otimes \beta)\Big). \tag{4.33}$$

Let us first compute the left-hand side of (4.33). Thus, $\varphi(a \otimes \alpha)\varphi(b \otimes \beta)$

$$\begin{aligned}
&= \Big[\big(a_{[0]} \otimes \alpha_{[0]}\big) \otimes \overline{a}_{[1]} S\overline{\alpha}_{[1]}\Big]\Big[\big(b_{[0]} \otimes \beta_{[0]}\big) \otimes \overline{b}_{[1]} S\overline{\beta}_{[1]}\Big] \\
&= \big(a_{[0]} \otimes \alpha_{[0]}\big)\big(b_{[0]} \otimes \beta_{[0]}\big) \otimes \overline{a}_{[1]} S\overline{\alpha}_{[1]} \overline{b}_{[1]} S\overline{\beta}_{[1]} \\
&= \big(a_{[0]} \otimes \alpha_{[0]}\big)\big(b_{[0]} \otimes \beta_{[0]}\big) \otimes \overline{a}_{[1]} S\overline{\alpha}_{[1]} \overline{b}_{[1]} S\overline{\beta}_{[1]} \\
&= a_{[0]2} b_{[0]} S\overline{a}_{[0]1[1]} \otimes \alpha_{[0]}\beta_{[0]}\big(S^{-1} a_{[0]3} _ a_{[0]1[0]}\big) \otimes \overline{a}_{[1]} S\overline{\alpha}_{[1]} \overline{b}_{[1]} S\overline{\beta}_{[1]} \\
&= a_{2[0]} b_{[0]} S\overline{a}_{1[0][1]} \otimes \alpha_{[0]}\beta_{[0]}\big(S^{-1} a_{3[0]} _ a_{1[0][0]}\big) \otimes \overline{a}_{1[1]}\overline{a}_{2[1]}\overline{a}_{3[1]} S\overline{\alpha}_{[1]} \overline{b}_{[1]} S\overline{\beta}_{[1]}
\end{aligned}$$

We have used the following identity:

$$x_{1[0]} \otimes x_{2[0]} \otimes x_{3[0]} \otimes x_{[1]} = x_{[0]1} \otimes x_{[0]2} \otimes x_{[0]3} \otimes \overline{x}_{1[1]}\overline{x}_{2[1]}\overline{x}_{3[1]}.$$

Now computing the right-hand side of (4.33), we have $\varphi\Big((a \otimes \alpha)(b \otimes \beta)\Big)$ :

$$\begin{aligned}
&= \varphi\Big[a_2 b\overline{a}_{1[1]} \otimes \alpha\beta\big(S^{-1}a_3 _ a_{1[0]}\big)\Big] \\
&= a_{2[0]} b_{[0]} \big(S\overline{a}_{1[1]}\big)_{[0]} \otimes \alpha_{[0]}\beta\big(S^{-1}a_3 _ a_{1[0]}\big)_{[0]} \otimes \overline{a}_{2[1]}\overline{b}_{[1]}\big(S\overline{a}_{1[1]}\big)_{[1]} S\overline{\alpha}_{[1]} S\overline{\beta}_{[1]} \big(S^{-1}\overline{a}_3\big)_{[1]}\overline{a}_{1[0][1]} \\
&= a_{2[0]} b_{[0]} S\overline{a}_{1[1][0]} \otimes \alpha_{[0]}\beta_{[0]}\big(S^{-1}a_{3[0]} _ a_{1[0][0]}\big)_{[0]} \otimes \overline{a}_{2[1]}\overline{b}_{[1]}\overline{a}_{1[1][1]} S\overline{\alpha}_{[1]} S\overline{\beta}_{[1]}\overline{a}_{3[1]}\overline{a}_{1[0][1]}
\end{aligned}$$

Now since the crossing is an $H$-comodule morphism, the commutativity of the following diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & M \otimes H \arrow[d] \\
M \otimes H \arrow[r] & (M \otimes H) \otimes H
\end{tikzcd}$$

gives the following identity:

$$x_{[0][0]} \otimes \overline{x}_{[0][1]} \otimes \overline{x}_{[1]} = x_{[0][0]} \otimes \overline{x}_{[1][0]} \otimes x_{[1][1]}\overline{x}_{[0][1]}$$

Hence we have,

$$x_{[0][0]} \otimes S\Big(\overline{x}_{[0][1]}\Big) \otimes \overline{x}_{[1]} = x_{[0][0]} \otimes S\Big(\overline{x}_{[1][0]}\Big) \otimes x_{[1][1]}\overline{x}_{[0][1]},$$

which makes both the sides of equation (4.33) equal. □

Next we discuss the antipode for $D(A)$. It is given by the following formula:

$$S(a \otimes \alpha) = \big(Sa_{[0]}\overline{a}_{[1]} \otimes 1\big)\big(1 \otimes S\alpha\big). \tag{4.34}$$

**Lemma 4.4.6** *The map given by equation (4.34) satisfies the antipode axiom.*

Proof: The map given by equation (4.34) will satisfy the antipode axiom if and only if for any $a \in A$, and $\alpha \in A^{\circ}$, following holds:

$$S\Big((a \otimes \alpha)_1\Big)(a \otimes \alpha)_2 = (a \otimes \alpha)_1 S\Big((a \otimes \alpha)_2\Big) = \epsilon(a)\alpha(1).$$

That is,

$$S\Big((a_{1[0]} \otimes \alpha_2)\Big)(S\overline{a}_{1[1]}a_2 \otimes \alpha_1) = (a_{1[0]} \otimes \alpha_2)S\Big(S\overline{a}_{1[1]}a_2 \otimes \alpha_1\Big) = \epsilon(a)\alpha(1). \tag{4.35}$$

Let us first compute the extreme left-hand side of (4.35). Then, $S\Big((a_{1[0]}\otimes\alpha_2)\Big)(S\overline{a}_{1[1]}a_2\otimes \alpha_1)$

$$\begin{aligned}
&= \Big(Sa_{1[0][0]}\overline{a}_{1[0][1]} \otimes 1\Big)\Big(1 \otimes S\alpha_2\Big)\Big(S\overline{a}_{1[1]}a_2 \otimes \alpha_1\Big)\\
&= \Big(Sa_{1[0][0]}\overline{a}_{1[0][1]} \otimes 1\Big)\Big(1 \otimes S\alpha_2\Big)\Big(S\overline{a}_{1[1]}a_2 \otimes 1)\big(1 \otimes \alpha_1\big)\\
&= \Big(Sa_{1[0][0]}\overline{a}_{1[0][1]} \otimes 1\Big)\Big(S\overline{a}_{1[1]}a_2 \otimes S\alpha_2)\big(1 \otimes \alpha_1\big)\\
&= \Big(Sa_{1[0][0]}\overline{a}_{1[0][1]}S\overline{a}_{1[1]}a_2 \otimes S\alpha_2\Big)\big(1 \otimes \alpha_1\big)\\
&= \Big(Sa_{1[0]}\overline{a}_{1[1]1}S\overline{a}_{1[1]2}a_2 \otimes S\alpha_2\Big)\big(1 \otimes \alpha_1\big) && \big(\text{ using, } x_{[0][0]} \otimes \overline{x}_{[0][1]} \otimes \overline{x}_{[1]} = x_{[0]} \otimes \overline{x}_{[1]1} \otimes \overline{x}_{[1]2}\big)\\
&= \Big(Sa_{1[0]}\epsilon(\overline{a}_{1[1]})a_2 \otimes S\alpha_2\Big)\big(1 \otimes \alpha_1\big) && \big(\text{ using, } x_1Sx_2 = Sx_1x_2 = \epsilon(x)\big)\\
&= \Big(S(a_1)a_2 \otimes S\alpha_2\Big)\big(1 \otimes \alpha_1\big) && \big(\text{ using, } x_{[0]}\epsilon(x_{[1]}) = x\big)\\
&= \big(\epsilon(a) \otimes S\alpha_2\big)\big(1 \otimes \alpha_1\big)\\
&= \epsilon(a)\alpha(1) && \big(\text{ using, } S\alpha_2\alpha_1 = \epsilon(\alpha) = \alpha(1)\big).
\end{aligned}$$

Computing the second term of (4.35) gives $(a_{1[0]}\otimes\alpha_2)S\big(S\overline{a}_{1[1]}a_2\otimes\alpha_1\big)$

$$
\begin{aligned}
&= (a_{1[0]}\otimes\alpha_2)\Big(S\big(S\overline{a}_{1[1]}a_2\big)_{[0]}\overline{\big(S\overline{a}_{1[1]}a_2\big)}_{[1]}\otimes 1\Big)\big(1\otimes S\alpha_1\big)\\
&= (a_{1[0]}\otimes\alpha_2)\Big(\overline{a}_{1[1][0]}Sa_{2[0]}\overline{a}_{1[1][1]}\overline{a}_{2[1]}\otimes 1\Big)\big(1\otimes S\alpha_1\big)\\
&= (a_{1[0]}\otimes 1)(1\otimes\alpha_2)\Big(\overline{a}_{1[1][0]}Sa_{2[0]}\overline{a}_{1[1][1]}\overline{a}_{2[1]}\otimes 1\Big)\big(1\otimes S\alpha_1\big)\\
&= (a_{1[0]}\otimes 1)\Big(\overline{a}_{1[1][0]}Sa_{2[0]}\overline{a}_{1[1][1]}\overline{a}_{2[1]}\otimes\alpha_2\Big)\big(1\otimes S\alpha_1\big)\\
&= \Big(a_{1[0]}\overline{a}_{1[1][0]}Sa_{2[0]}\overline{a}_{1[1][1]}\overline{a}_{2[1]}\otimes\alpha_2\Big)\big(1\otimes S\alpha_1\big)\\
&= \Big(a_{1[0]}\,\overline{a}_{1[1]}\,Sa_{2[0]}\,\overline{a}_{2[1]}\otimes\alpha_2\Big)\big(1\otimes S\alpha_1\big) \qquad \big(\text{ using } \overline{x}_{[0]}\overline{x}_{[1]}=\overline{x}\big)\\
&= \Big(a_{1[0]}\,Sa_{2[0]}\,\overline{a}_{1[1]}\,\overline{a}_{2[1]}\otimes\alpha_2\Big)\big(1\otimes S\alpha_1\big)\\
&= \Big(\big(a_1\,Sa_2\big)_{[0]}\,\overline{\big(a_1\,Sa_2\big)}_{[1]}\otimes\alpha_2\Big)\big(1\otimes S\alpha_1\big)\\
&= \Big(\varphi\big(\epsilon(a)\big)\otimes\alpha_2\Big)\big(1\otimes S\alpha_1\big)\\
&= \epsilon(a)\alpha(1) \qquad \big(\text{ using } \alpha_2 S\alpha_1=\epsilon(\alpha)=\alpha(1)\big).
\end{aligned}
$$

□

Perhaps, the quasicoherent sheaf $\widetilde{D(A)}$ of $\mathcal{O}_{\mathrm{Spec}\,(H)}$-modules, generated by the $\mathbb{K}$-module $D(A)$ has a crossed Hopf $\mathcal{G}$-coalgebra structure on it. Thus, we have the following theorem.

**Theorem 4.4.7** *$D(\mathcal{A})=\widetilde{D(A)}$ is a crossed Hopf $\mathcal{G}$-coalgebra.*

Proof: Let us specialise $D(\mathcal{A})$ to $\mathbb{G}=\mathcal{G}(\mathbb{S})$. Let us denote the specialisation at $\mathbb{S}$ by $D(\mathcal{A})_{\mathbb{S}}$. For any $g\in\mathbb{G}$, let $D_{g^{-1}}=\mathbb{S}\otimes_g D(A)=(\mathbb{S}\otimes_g A)\otimes A^{\circ}=A_g\otimes A^{\circ}$. Then it suffices to show that $D(\mathcal{A})$ specialises to a crossed Hopf $\mathbb{G}$-coalgebra, $D(\mathcal{A})_{\mathbb{S}}$. Observe that Lemma 4.4.3 to Lemma 4.4.6 together shows that $D(A)$ forms a Hopf algebra over $\mathbb{K}$ with a crossing. Let us now show that the four fundamental operations of $D(A)$ namely, multiplication, comultiplication, antipode and crossing, are infact a global version of the corresponding operations of Zunino's quantum double.

(i) For each $a\otimes\alpha\in D(A)$, let $\widetilde{a}\otimes\alpha=(1\otimes_{\mathbb{S}}a)\otimes\alpha$ be the corresponding specialisation.

So, for any $\widetilde{a}\otimes\alpha$, $\widetilde{b}\otimes\beta\in D_{g^{-1}}$

$$\begin{aligned}(\widetilde{a}\otimes\alpha)(\widetilde{b}\otimes\beta) &= \widetilde{a}_2 b\widetilde{S\overline{a}_{1[1]}}\otimes\alpha\beta\big(\widetilde{S^{-1}a}_{3\_}\widetilde{a}_{1[0]}\big)\\ &= \widetilde{a}_2 b\otimes\alpha\beta\big(S^{-1}\widetilde{a}_{3\_}\widetilde{S\overline{a}_{1[1]}}\widetilde{a}_{1[0]}\big)\\ &= \widetilde{a}_2 b\otimes\alpha\beta\big(S^{-1}\widetilde{a}_{3\_}\varphi_g(\widetilde{a}_1)\big)\end{aligned}$$

which is a global version of (4.20) i.e. sum over $f,h\in\mathbb{G}$ where $\widetilde{a}\mapsto\widetilde{a}_1\otimes\widetilde{a}_2\otimes\widetilde{a}_3$ for $\widetilde{a}_1\in A_{ghg^{-1}}$, $\widetilde{a}_2\in A_g$, $\widetilde{a}_3\in A_{h^{-1}}$ and $\alpha\in A^\circ_f$. Clearly $\beta\in A^\circ_h$.

(ii) Let $\widetilde{a}\otimes\alpha\in D_{hg}$. This implies, $\widetilde{a}\otimes\alpha\in A_{(hg)^{-1}}\otimes A^\circ$. Let $\widetilde{a}\mapsto\widetilde{a}_1\otimes\widetilde{a}_2$ for $\widetilde{a}_1\in A_{g^{-1}h^{-1}g}$ and $\widetilde{a}_2\in A_{g^{-1}}$, then

$$\begin{aligned}\Delta(\widetilde{a}\otimes\alpha) &= \Big(\widetilde{a}_{1[0]}\otimes\alpha_2\Big)\otimes\Big(\widetilde{S\overline{a}_{1[1]}}a_2\otimes\alpha_1\Big)\\ &= \Big(\widetilde{a}_{1[0]}\widetilde{S\overline{a}_{1[1]}}\otimes\alpha_2\Big)\otimes\Big(a_2\otimes\alpha_1\Big)\\ &= \Big(\varphi_g(\widetilde{a}_1)\otimes\alpha_2\Big)\otimes\Big(a_2\otimes\alpha_1\Big)\end{aligned}$$

which is again a global version of (4.23)

(iii) Let $\widetilde{a}\otimes\alpha\in D_h$. This implies, $\widetilde{a}\otimes\alpha\in A_{h^{-1}}\otimes A^\circ$. For $g\in\mathbb{G}$ consider,

$$\begin{aligned}\varphi_g(\widetilde{a}\otimes\alpha) &= \Big(\widetilde{a}_{[0]}\otimes\alpha_{[0]}\Big)\otimes\widetilde{\overline{a}}_{[1]}S\overline{\alpha}_{[1]}\\ &= \widetilde{a}_{[0]}\widetilde{\overline{a}}_{[1]}\otimes\alpha_{[0]}S\alpha_{[1]}\\ &= \varphi_g a\otimes\varphi^*_{g^{-1}}\alpha,\end{aligned}$$

which is a global version of (4.26)

(iv) Let $\widetilde{a}\otimes\alpha\in D_g$. Consider,

$$\begin{aligned}S_g(\widetilde{a}\otimes\alpha) &= \Big(\widetilde{Sa}_{[0]}\widetilde{\overline{a}}_{[1]}\otimes 1\Big)\Big(1\otimes S\alpha\Big)\\ &= \Big(\widetilde{(Sa)}_{[0]}\widetilde{(S\overline{a})}_{[1]}\otimes 1\Big)\Big(1\otimes S\alpha\Big)\\ &= \Big(\varphi_g S_{g^{-1}}\widetilde{a}\otimes 1\Big)\Big(1\otimes S\alpha\Big)\end{aligned}$$

which is a global version of (4.25). Once we have shown the equivalence of all the formulaes to the one given by Zunino, we can use the result established by him to say that $D(\mathcal{A})_{\mathbb{S}}$ is a Hopf $\mathbb{G}$-coalgebra with values in $\mathbb{S}$ which completes the proof.

$\square$

Note that $D(\mathcal{A})$ is topologically quasitriangular in general. This is so because

$$R \in (A \otimes A^{\circ})\widehat{\otimes}(A \otimes A^{\circ})$$

The $R$-matrix is in the completion of the tensor above.

Our goal is to establish a morphism between the quantum double $D_Z(\mathcal{A}_{\mathbb{S}})$ given by Zunino, and $D(\mathcal{A})_{\mathbb{S}}$ and thus interpret $D(\mathcal{A})_{\mathbb{S}}$ as a quotient of $D_Z(\mathcal{A}_{\mathbb{S}})$.

Now once we have a Hopf $\mathcal{G}$-coalgebra, we can have its specialisation at $\mathbb{S}$ giving us a Hopf $\mathbb{G}$-coalgebra where $\mathbb{G} = \mathcal{G}(\mathbb{S})$. We denote the specialisation of $D(\mathcal{A})$ at $\mathbb{S}$ by $D(\mathcal{A})_{\mathbb{S}}$.

Let $\mathbb{S}$ be a finite dimensional commutative $\mathbb{K}$-algebra. For every $f \in \mathbb{G}$, there is a natural map of $\mathbb{K}$-algebras

$$\theta_f : A \longrightarrow \mathbb{S} \otimes_f A = A \to A_f.$$

Note that $\{\theta_f\}_f$ is a collection of morphisms from coalgebra $A$ to the $\mathcal{G}(\mathbb{S})$-coalgebra $\{\mathbb{S}\otimes_f A\}_f$. Further, since tensor product is a covariant functor, we have a homomorphism of $\mathbb{S}$-algebras

$$1 \otimes \theta_f : \mathbb{S} \otimes_{\mathbb{K}} A \longrightarrow \mathbb{S} \otimes_{\mathbb{K}} \mathbb{S} \otimes_f A.$$

Then using multiplication $m_{\mathbb{S}}$ of $\mathbb{S}$, we have the following composition of $\mathbb{S}$-algebra homomorphism

$$\zeta_f : \mathbb{S} \otimes_{\mathbb{K}} A \xrightarrow{1\otimes\theta_f} \mathbb{S} \otimes_{\mathbb{K}} \mathbb{S} \otimes_f A \xrightarrow{m_{\mathbb{S}}\otimes 1_A} \mathbb{S} \otimes_f A = A_f.$$

Now taking finite duals with values in $\mathbb{S}$, we get a map of $\mathbb{S}$-coalgebras

$$\zeta_f^{\circ} : A_f^{\circ} \longrightarrow \big(\mathbb{S} \otimes_{\mathbb{K}} A\big)^{\circ}. \tag{4.36}$$

**Lemma 4.4.8** *There exists an isomorphism of* $\mathbb{S}$*-algebras* $\big(\mathbb{S} \otimes_{\mathbb{K}} A\big)^{\circ} \cong_{\mathbb{S}} \mathbb{S} \otimes_{\mathbb{K}} A^{\circ}$.

PROOF: By definition,

$$\big(\mathbb{S} \otimes_{\mathbb{K}} A\big)^{\circ} = \mathrm{Hom}^{\circ}_{\mathbb{S}}(\mathbb{S} \otimes_{\mathbb{K}} A, \mathbb{S}) \tag{4.37}$$

where $\mathrm{Hom}^{\circ}_{\mathbb{S}}$ is an $\mathbb{S}$-algebra of all the $\mathbb{S}$-linear maps from $\mathbb{S} \otimes_{\mathbb{K}} A$ to $\mathbb{S}$ that vanish on some finite coprojective ideal of $\mathbb{S} \otimes_{\mathbb{K}} A$. Consider the natural map,

$$\varphi : \mathbb{S} \otimes_{\mathbb{K}} A^{\circ} \to \mathrm{Hom}_{\mathbb{S}}\big(\mathbb{S} \otimes_{\mathbb{K}} A, \mathbb{S}\big) \text{ given by}$$

$$\varphi(s \otimes \alpha) : t \otimes a \mapsto st\alpha(a),$$

where $s, t \in \mathbb{S}$ and $\alpha \in A^{\circ}$. Then $\varphi$ is clearly an injective $\mathbb{S}$-algebra morphism. For $\mathrm{Im}\,\varphi \subseteq \mathrm{Hom}^{\circ}_{\mathbb{S}}(\mathbb{S} \otimes_{\mathbb{K}} A, \mathbb{S})$ it suffice to show that for any $s \in \mathbb{S}$ and $\alpha \in A^{\circ}$, $\varphi(s \otimes \alpha) \in \mathrm{Hom}^{\circ}_{\mathbb{S}}(\mathbb{S} \otimes_{\mathbb{K}} A, \mathbb{S})$. Let $\alpha \in A^{\circ}$ such that $\alpha(I) = 0$ and $\dim_{\mathbb{K}}(A/I) < \infty$. Then,

$$\alpha : A \xrightarrow{-} A/I \xrightarrow{\overline{\alpha}} K \tag{4.38}$$

$\alpha$ factors through a finite dimensional $\mathbb{K}$-algebra $(A/I)$. Using above equation, we have:

$$\varphi(s \otimes \alpha) : \mathbb{S} \otimes A \to \mathbb{S} \otimes A/I \to S$$

where $\mathbb{S} \otimes A/I \cong \big(\mathbb{S} \otimes A\big)/\big(\mathbb{S} \otimes I\big)$. Since $\mathbb{S}$ is a finite dimensional $\mathbb{K}$-algebra and $\dim_{\mathbb{K}}(A/I) < \infty$, implies that $\big(\mathbb{S} \otimes A\big)/\big(\mathbb{S} \otimes I\big)$ is finite dimensional. Hence $\varphi(s \otimes \alpha) \in \mathrm{Hom}^{\circ}_{\mathbb{S}}(\mathbb{S} \otimes_{\mathbb{K}} A, \mathbb{S})$. Conversely, let $\beta \in \mathrm{Hom}^{\circ}_{\mathbb{S}}(\mathbb{S} \otimes A, \mathbb{S})$ such that $\beta(J) = 0$ for some finite coprojective ideal $J$ of $\mathbb{S} \otimes_{\mathbb{K}} A$. Then,

$$\beta : \mathbb{S} \otimes_{\mathbb{K}} A \to \big(\mathbb{S} \otimes_{\mathbb{K}} A\big)/J \xrightarrow{\overline{\beta}} S.$$

Let $\{x^i, x_i\}$ be the dual $\mathbb{S}$-basis for $\big(\mathbb{S} \otimes A\big)/J$ and $\{e^j, e_j\}$ be the dual $\mathbb{K}$-basis for $\mathbb{S}$. Then,

$$\beta(s \otimes a) = \overline{\beta}(s \otimes a + J)$$

where

$$\overline{\beta} = \sum_i s_i x^i$$

for $s_i \in \mathbb{S}$. Let

$$\widetilde{x}^i : \mathbb{S} \otimes_{\mathbb{K}} A \to \big(\mathbb{S} \otimes_{\mathbb{K}} A\big)/J \xrightarrow{x^i} \mathbb{S}.$$

Then observe that

$$\beta = \varphi\big(\sum_{i,j} s_i e_j \otimes e^j \widetilde{x}^i\big).$$

Indeed,

$$\begin{aligned}
\varphi\big(\sum_{i,j} s_i e_j \otimes e^j \widetilde{x}^i\big)(t \otimes a) &= \sum_{i,j} t s_i e_j e^j(\widetilde{x}^i(a)) \\
&= \sum_{i} t s_i(\widetilde{x}^i(a)) \\
&= t\beta(a) \\
&= \beta(t \otimes a)
\end{aligned}$$

Thus, $\beta \in \operatorname{Im}\varphi$, which completes the proof. □

Let $\Gamma = \big(A^\circ\big)^{\mathrm{tot,cop}} = \bigoplus_{f\in\mathbb{G}} A_f^\circ$. Then by definition of $(\_)^{\mathrm{tot}}$ (defined by Zunino), it has a Hopf algebra structure. Here $\Gamma$ has a $\mathbb{S}$-Hopf algebra structure as each of the fibre $A_f^\circ$ is an $\mathbb{S}$-coalgebra. If $\varphi : A \to A \otimes H$ is the crossing of $A$ and $\varphi^\circ : A^\circ \to A^\circ \otimes H$ is the crossing of $A^\circ$, then the crossing of $\Gamma$ is simply the sum of the crossings of each of the fiber $A_f$. Let us denote it as $\varphi_\Gamma$ which is an $\mathbb{S}$-algebra morphism from $\Gamma$ to $\Gamma \otimes H$. Also, $1 \otimes \varphi^\circ : \mathbb{S} \otimes_{\mathbb{K}} A^\circ \longrightarrow \mathbb{S} \otimes_{\mathbb{K}} A^\circ \otimes H$ gives a crossing of $\mathbb{S} \otimes_{\mathbb{K}} A^\circ$.

Now using the canonical projection $\widetilde{\pi} : \bigoplus_{f\in\mathbb{G}} A_f^\circ \longrightarrow A_f^\circ$, for $f \in \mathbb{G}$ along with the above lemma and equation (4.36), we get the following map of $\mathbb{S}$-coalgebras:

$$\pi : \Gamma \longrightarrow \mathbb{S} \otimes_{\mathbb{K}} A^\circ$$

where $\pi = \zeta_f^\circ \widetilde{\pi}$ such that, $\pi(\alpha_f) : 1 \otimes_{\mathbb{S}} a \mapsto \alpha_f(a)$, $\alpha_f \in A_f^\circ$ for some $f \in \mathbb{G}$ and $a \in A$. Note that the way $\pi$ is defined, it preserves the crossing of the two $\mathbb{S}$-Hopf algebras $\Gamma$ and $\mathbb{S} \otimes_{\mathbb{K}} A^\circ$, i.e.

$$(1 \otimes \varphi^\circ)\pi = (\pi \otimes 1)\varphi_\Gamma. \tag{4.39}$$

Note that $A_{g^{-1}}$ is flat as an $\mathbb{S}$-algebra for any $g \in \mathbb{G}$. Thus for each $g \in \mathbb{G}$, we have following composition of maps:

$$\phi_{g^{-1}} : A_{g^{-1}} \otimes \Gamma \xrightarrow{1\otimes\pi} A_{g^{-1}} \otimes \mathbb{S} \otimes A^{\circ} \longrightarrow A_{g^{-1}} \otimes A^{\circ}.$$

Let us denote the collection of these maps by $\Phi = \{\phi_{g^{-1}}\}_{g\in\mathbb{G}}$. So far $\Phi$ is a collection of linear maps. We claim that this will be indeed a crossed Hopf group coalgebra morphism between the global sections of the two quantum doubles. Thus we have our main theorem of the chapter:

**Theorem 4.4.9** *Let $\mathcal{G} = \mathrm{Spec}\,(H)$ be an affine group scheme, $\mathcal{A}$ be a coherent, locally free crossed Hopf $\mathcal{G}$-coalgebra over $\mathcal{G}$, and $\mathbb{S}$ be a finite dimensional commutative $\mathbb{K}$-algebra. Then $\Phi$ is a morphism of crossed Hopf $\mathcal{G}(\mathbb{S})$-coalgebra from Zunino's quantum double $D_Z(\mathcal{A}_{\mathbb{S}})$ to $D(\mathcal{A})_{\mathbb{S}}$.*

Proof: For each $g \in \mathbb{G}$,

$$\phi_g : A_g \otimes \Big(\bigoplus_{f\in\mathbb{G}} A_f{}^{\circ}\Big) \longrightarrow A_g \otimes A^{\circ},$$

where $\Phi = \{\phi_{g^{-1}}\}_{g\in\mathbb{G}}$. Now $\Phi$ is a crossed Hopf $\mathcal{G}(\mathbb{S})$-coalgebra morphism if and only if it is compatible with respect to the four fundamental structures, namely multiplication, comultiplication, counit and crossing. Compatibility with respect to unit and antipode would follow automatically. Observe that

$$\Phi(1\otimes\alpha)(1\otimes\beta) = \Phi(1\otimes\alpha)\Phi(1\otimes\beta)$$

because $\pi$ is an algebra map which is true since $\theta_f$ intertwines coalgebra structure on $A$ with $\mathcal{G}(\mathbb{S})$-coalgebra structure on $\{A_f\}_f$, that is,

$$\Delta_{f,g} \circ \theta_{fg} = (\theta_f \otimes \theta_g) \circ \Delta.$$

Moreover, each $\phi_f$ preserves multiplication. Note that on more general elements in $D_Z(\mathcal{A}_{\mathbb{S}})$, $(a_g \otimes \alpha_f)(b_g \otimes \beta_h) = (a_g^{''} b_g \otimes \sum_{f,h} \alpha_f \beta_h \Big(\overline{S}_h(a_{h^{-1}}^{'''})\_\varphi_{g^{-1}}(a_{ghg^{-1}}^{'})\Big)$ where $a_g, b_g \in$

$A_g$, $\alpha_f \in \mathcal{A}^\circ_f$, $\beta_h \in \mathcal{A}^\circ_h$. Thus we have $\phi_g\Big[(a_g \otimes \alpha_f)(b_g \otimes \beta_h)\Big]$

$$\begin{aligned}
&= \phi_g\Big[(a''_g b_g \otimes \alpha_f \beta_h\Big(\overline{S}_h(a'''_{h^{-1}})_\varphi_{g^{-1}}(a'_{ghg^{-1}})\Big)\Big] \\
&= a''_g b_g \otimes \pi\Big[\alpha_f \beta_h\Big(\overline{S}_h(a'''_{h^{-1}})_\varphi_{g^{-1}}(a'_{ghg^{-1}})\Big)\Big] \\
&= a''_g b_g \otimes \pi\alpha_f \pi\Big[\beta_h\Big(S_h^{-1}(a'''_{h^{-1}})_\varphi_{g^{-1}}(a'_{ghg^{-1}})\Big)\Big] \qquad \text{(because } \pi \text{ is an algebra morphism)} \\
&= a''_g b_g \otimes \pi\alpha_f \pi\beta_h\Big(S_h^{-1}(a'''_{h^{-1}})_\varphi_{g^{-1}}(a'_{ghg^{-1}})\Big) \qquad \text{(because } \pi \text{ is a coalgebra morphism.)}
\end{aligned}$$

where $a_g \mapsto a'_{ghg^{-1}} \otimes a''_g \otimes a'''_{h^{-1}}$. Compare it with the multiplication of $\phi_g\big[(a_g \otimes \alpha_f)\big]$ and $\phi_g\big[(b_g \otimes \beta_h)\big]$ in $D(\mathcal{A})_{\mathbb{S}}$. In the latter case we use Theorem (4.4.7) and we have:

$$\begin{aligned}
&= \big[a_g \otimes \pi\alpha_f\big]\big[b_g \otimes \pi\beta_h\big] \\
&= a''_g b_g \otimes \pi\alpha_f \pi\beta_h\Big(S^{-1}(a'''_{h^{-1}})_\varphi_{g^{-1}}(a'_{ghg^{-1}})\Big).
\end{aligned}$$

Thus multiplication is preserved under $\Phi$. Similarly we can show that $\Phi$ preserves the other operations. Calculations are quite similar as above or as we have done in Theorem 4.4.7. In this way, $\Phi$ becomes indeed a morphism of crossed Hopf $\mathbb{G}$-coalgebra. This completes the proof.

□

Observe that in case the maps $\theta_f$ are injections, then $\pi$ in turn becomes a surjection, making each of $\phi_f$ a surjection. In such a case $D(\mathcal{A})_{\mathbb{S}}$ becomes a quotient of Zunino's quantum double $D_Z(\mathcal{A}_{\mathbb{S}})$.

**Example.** Quantum double in affine case.

Assume that $0 \neq q \in \mathbb{K}$ and that $q$ is not a root of unity. Let $\tilde{A}$ be the coordinate ring of quantum $2 \times 2$ matrices, which is defined to be $\tilde{A} = \mathcal{O}_q\big(M_2(\mathbb{K})\big) = \mathbb{K}\langle a, b, c, d\rangle$ subject to the relations

$$ba = q^{-2}ab \qquad ca = q^{-2}ac \qquad bc = cb$$

$$db = q^{-2}bd \qquad dc = q^{-2}cd \qquad ad - da = (q^2 - q^{-2})bc$$

Then $\tilde{A}$ becomes a bialgebra by the following setting. If we write

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \tag{4.40}$$

the coproduct and counit are given by $\Delta(X_{ij}) = \sum_k X_{ik} \otimes X_{kj}$, and $\epsilon(X_{ij}) = \delta_{ij}$. Let the quantum determinant be $\det_q X = ad - q^2 bc$. Let $A$ be the quantum $SL_2(\mathbb{K})$ is defined as

$$\mathcal{O}_q(SL_2(\mathbb{K})) = \mathcal{O}_q(M_2(\mathbb{K}))/(\det_q X - 1).$$

It inherits a bialgebra structure from $\mathcal{O}_q(M_2(\mathbb{K}))$. Indeed, it forms a Hopf algebra structure with the antipode map given as

$$S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (ad - q^2 bc)^{-1} \begin{bmatrix} d & -q^{-2} b \\ -q^2 c & a \end{bmatrix}$$

That is, $Sa = (ad - q^2 bc)^{-1} d, etc.$ Now let $B$ be the $q$-analogue of the universal enveloping algebra of the classical Lie algebra $\mathfrak{g} = sl(2)$. Let us recall the definition.

$$B = U_q\big(sl(2)\big) = \mathbb{K}\langle E, F, K, K^{-1}\rangle$$

with the relations $KE = q^2 EK$, $KF = q^{-2} FK$ and $EF - FE = \dfrac{K^2 - K^{-2}}{(q^2 - q^{-2})}$. There is a Hopf algebra structure on $B$ defined by

$$\Delta(E) = E \otimes K^{-1} + K \otimes E, \quad S(E) = -q^{-2} E, \quad \epsilon(E) = 0$$

$$\Delta(F) = F \otimes K^{-1} + K \otimes F, \quad S(F) = -q^{2} F, \quad \epsilon(F) = 0$$

$$\Delta(K) = K \otimes K, \quad S(K) = K^{-1}, \quad \epsilon(K) = 1$$

Both $\Delta$ and $\epsilon$ extend to algebra homomorphisms, and $S$ extends to an algebra anti-homomorphism.

Let us provide a pairing $\sigma : A \otimes B \to \mathbb{K}$ between the two Hopf algebras $A$ and $B$

as follows : where $\sigma(a,E) = 0$, $\sigma(b,E) = 1$, etc. One can easily check that equations

| | $E$ | $F$ | $K$ | $K^{-1}$ |
|---|---|---|---|---|
| $a$ | 0 | 0 | $q$ | $q^{-1}$ |
| $b$ | 0 | 1 | 0 | 0 |
| $c$ | 1 | 0 | 0 | 0 |
| $d$ | 0 | 0 | $q^{-1}$ | $q$ |

(2.11)-(2.13) all hold true on the set of generators of $A$ and $B$, making $\sigma$ a Hopf pairing.

Now let $\mathcal{G}$ be one-dimensional torus $\mathbb{K}^{\times}$. Then for any $\alpha \in \mathcal{G}$, we define an endomorphism $\phi_\alpha : B \to B$ which maps $E$ and $F$ to $\alpha^2 E$ and $\alpha^{-2}F$ respectively without affecting $K$. Clearly, $\phi_\alpha$ is a Hopf algebra endomorphism of $A$. Set $D(A, B, \sigma, \phi_\alpha)$ as the tensor product of the two $\mathbb{K}$-spaces $A$ and $B$. Then by Section(2.1) in [Vir05], $D(A, B, \sigma, \phi_\alpha)$ has a structure of an associative and unitary algebra given by

$$(a \otimes b) \otimes (a^{'} \otimes b^{'}) = \sigma\Big(\phi(a^{'}_{(1)}, S(b_{(1)}))\Big)\sigma\Big(a^{'}_{(3)}, b_{(3)})\Big)aa^{'}_{(2)} \otimes b_{(2)}b^{'},$$

$$1_{D(A,B;\sigma,\phi)} = 1_A \otimes 1_B$$

for any $a, a^{'} \in A$ and $b, b^{'} \in B$. Theorem(2.3) in [Vir05] says that the family of algebras $D(A, B, \sigma, \phi) = \{D(A, B, \sigma, \phi_\alpha)\}_{\alpha\in\mathcal{G}}$ has a Hopf $\mathcal{G}$-coalgebra structure given by :

$$\Delta_{\alpha,\beta}(a \otimes b) = \Big(\phi_\beta(a_{(1)}) \otimes b_{(1)}\Big) \otimes \Big(a_{(2)} \otimes b_{(2)}\Big),$$

$$\epsilon(a \otimes b) = \epsilon_A(a)\epsilon_B(b),$$

$$S_\alpha(a \otimes b) = \sigma\Big(\phi_\alpha(a_{(1)}), b_{(1)}\Big)\sigma\Big((a_{(3)}), S(b_{(3)})\Big)\ \Big[\phi_\alpha S(a_{(2)}) \otimes S(b_{(2)})\Big].$$

## 4.5 Quantum double in general case

In this section we sketch non-rigorously how quantum double works for an arbitrary group scheme. Let $\mathcal{A}$ be a Hopf $\mathcal{G}$-coalgebra. Let us recall from Chapter 2 the dual of $\mathcal{A}$ denoted by $\mathcal{A}^*$, given by:

$$\mathcal{A}^* = \mathrm{Hom}_{\mathbb{K}}(\mathcal{A}, \mathbb{K}).$$

Then for each open covering $\{U_i\}$ of $U \subseteq \mathcal{G}$, the sheaf exact sequence of $\mathbb{K}$-modules,

$$\prod_{i,j} \mathcal{A}(U_i \cap U_j) \leftleftarrows \prod_{i} \mathcal{A}(U_i) \longleftarrow \mathcal{A}(U) \longleftarrow 0$$

turns into cosheaf exact sequence of $\mathbb{K}$-modules

$$\oplus_{i,j}\mathcal{A}^*(U_i \cap U_j) \rightrightarrows \oplus_i \mathcal{A}^*(U_i) \longrightarrow \mathcal{A}^*(U) \longrightarrow 0.$$

Thus $\mathcal{A}^*$ is a cosheaf of $\mathbb{K}$-modules over group scheme $\mathcal{G}$. Now let us recall from chapter 2 the finite dual of $\mathcal{A}$. From (2.10), we have

$$\mathcal{A}^\circ = \Big\{ f \in \mathcal{A}^* \Big| f(I) = 0 \text{ for some ideal } I \text{ of } \mathcal{A} \text{ ; } \dim_{\mathbb{K}} \mathcal{A}/I < \infty \Big\}.$$

Equivalently, we can say that $\mathcal{A}^\circ$ has those elements of $\mathcal{A}^*$ which vanish on some finite coprojective ideal of $\mathcal{A}$. Cosections of $\mathcal{A}^\circ$ are subspaces of the corresponding cosections of $\mathcal{A}^*$. Indeed they are coalgebras over $\mathbb{K}$, (cf. (2.16)). For each inclusion of open sets $U \subseteq V$, the corestriction $cores^\circ_{V,U} : \mathcal{A}^\circ(U) \to \mathcal{A}^\circ(V)$ is given by the following sequence

$$cores^\circ_{V,U}(f) : \mathcal{A}(V) \xrightarrow{res_{U,V}} \mathcal{A}(U) \longrightarrow {}_f M \longrightarrow \mathbb{K},$$

Thus the corestrictions of the cosheaf $\mathcal{A}^\circ$ are induced by the restrictions of $\mathcal{A}^*$. This makes $\mathcal{A}^\circ$ a subcosheaf of $\mathcal{A}^*$.

Note that $\mu_*(\mathcal{A} \boxtimes \mathcal{A})(\mathcal{G}) \neq \mathcal{A}(\mathcal{G}) \otimes \mathcal{A}(\mathcal{G})$. Infact, $\mathcal{A}(\mathcal{G}) \otimes \mathcal{A}(\mathcal{G})$ sits naturally inside $\mu_*(\mathcal{A} \boxtimes \mathcal{A})(\mathcal{G})$. Consider the following example:

**Example.** Let $\mathcal{G}$ be a discrete infinite group. Let $\mathcal{A}$ be the constant sheaf over $\mathcal{G}$ with fibre at each point to be the ground field $\mathbb{K}$. Comultiplication on $\mathcal{A}$ is simply the identity map $\Delta_{g,h} : \mathcal{A}_{g,h} \to \mathcal{A}_g \otimes_{\mathbb{K}} \mathcal{A}_h$ as $\mathcal{A}_{gh} = \mathbb{K}$, and $\mathcal{A}_g \otimes \mathcal{A}_h = \mathbb{K} \otimes \mathbb{K} = \mathbb{K}$. $\mathcal{A}(\mathcal{G}) = \prod_{g \in \mathcal{G}} \mathcal{A}_g = \prod_{g \in \mathcal{G}} \mathbb{K}_g$. The global section is the infinite product over $\mathcal{G}$ of the copies of $\mathbb{K}$. If $x \in \mathcal{A}(\mathcal{G})$, then clearly, $\Delta x \in \prod_{g,h \in \mathcal{G}} (\mathcal{A}_g \otimes \mathcal{A}_h)$ but $\prod_{g \in \mathcal{G}} \mathcal{A}_g \otimes \prod_{h \in \mathcal{G}} \mathcal{A}_h \subsetneq \prod_{g,h \in \mathcal{G}} (\mathcal{A}_g \otimes \mathcal{A}_h)$. Now the dual cosheaf is given as $\mathcal{A}(\mathcal{G})^\circ = \{\sum_{i=i}^n \alpha_i \pi_{g_i} | \alpha_i \in \mathbb{K}\}$; $\pi_g : \mathcal{A}(\mathcal{G}) \to \mathcal{A}(\mathcal{G})$. Then $\mathcal{A}^\circ(\mathcal{G})$ is isomorphic to the group algebra, $\mathbb{K}\mathcal{G}$ (cf. Theorem 4.5.3).

Next we discuss the global section of $\mathcal{A}$. Let

$$\Gamma = \Gamma(\mathcal{G}, \mathcal{A}) = \mathcal{A}(\mathcal{G}).$$

Let us recall the result we proved in Chapter 2, Lemma (2.6.1):

**Lemma 4.5.1** *Suppose $\mathcal{A}$ is a sheaf of algebras over a group scheme $\mathcal{G}$. If $\mathcal{A}$ is generated by global sections $\Gamma$, then $\mathcal{A} \boxtimes \mathcal{A}$ is generated by $\Gamma \times \Gamma$.*

This helps us to formulate the following conjecture:

**Conjecture 4.5.2** *Let $U \subseteq \mathcal{G}$ be an open subset such that $\mu^{-1}(U) \subseteq \pi_1^{-1}(U) \cap \pi_2^{-1}(U)$ and $\operatorname{Im} e \subseteq U$. Let $\mathcal{A}$ be finitely generated by global sections. Then $\mathcal{A}^\circ(U)$ is an associative algebra with identity.*

In particular, the global cosection of the dual cosheaf $\mathcal{A}^\circ(\mathcal{G})$ is an associative algebra with identity. In fact a bigger statement is true which is stated in the form of a theorem below. We do not prove the theorem as this section is mainly speculative, and it uses the above conjecture.

**Conjecture 4.5.3** *Suppose $\mathcal{A}$ is a Hopf $\mathcal{G}$-coalgebra generated by global sections. Then $\mathcal{A}^\circ(\mathcal{G})$ is a Hopf algebra over $\mathbb{K}$ in the usual sense of a vector space.*

The above theorem is crucial as it assists in framing the structure of a quantum double in the general case.

**Conjecture 4.5.4** *Suppose $\mathcal{A}$ is a crossed Hopf $\mathcal{G}$-coalgebra generated by global sections. Then $D(\mathcal{A}) := \mathcal{A} \otimes \Gamma$ is a quasitriangular crossed Hopf $\mathcal{G}$-coalgebra, where $\Gamma = \mathcal{A}^\circ(\mathcal{G})$.*

The structures of $D(\mathcal{A})$ are consistent with Theorem (4.4.7). Further, if $\mathcal{A}$ is not generated by global sections, it should be a *tensor product* of a sheaf $\mathcal{A}$ and its dual cosheaf $\mathcal{A}^\circ$ which will be studied by the mathematicians of future generations.

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